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Editors' Remarks

From the Poem

THE BALLAD OF EAST AND WEST

****

Oh, East is East, and West is West, and never the twain shall meet,

Till Earth and Sky stand presently at God's great Judgment Seat;

But there is neither East nor West, Border, nor Breed, nor Birth,

When two strong men stand face to face,

tho' they come from the ends of the earth!

****

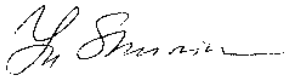
Rudyard Kipling[*], 1889

The 15th volume No.2 presents the papers on the actual topics such as **Applied Statistics and Operations Research**, **Mathematical Methods**, **Information Processing** and **Mathematical Modelling**.

Our journal policy is directed on the fundamental and applied sciences researches, which are the basement of a full-scale modelling in practice.

This edition is the continuation of our publishing activities. We expect our journal will be interesting for research community, and we are open for collaboration both in research and publishing. This number continues the current 2011 year of our publishing work. We hope that journal's contributors will consider the collaboration with the Editorial Board as useful and constructive.

EDITORS



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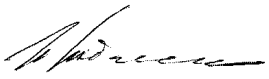

I.V. Kabashkin

[*] **Joseph Rudyard Kipling** (30 December 1865 – 18 January 1936) - an English poet, short-story writer, and novelist chiefly remembered for his celebration of British imperialism, tales and poems of British soldiers in India, and his tales for children. Kipling received the 1907 Nobel Prize for Literature. He was born in Bombay, in the Bombay Presidency of British India. Kipling is best known for his works of fiction, including *The Jungle Book* (a collection of stories which includes "Rikki-Tikki-Tavi"), *Just So Stories* (1902) (1894), *Kim* (1901) (a tale of adventure), many short stories, including "The Man Who Would Be King" (1888); and his poems, including *Mandalay* (1890), *Gunga Din* (1890), *The White Man's Burden* (1899).

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ADJUSTING MARKOV MODELS TO CHANGES IN MAINTENANCE POLICY FOR RELIABILITY ANALYSIS

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Equipment deterioration can be analysed with discrete state-transition models which are used as a foundation for numerical evaluation of various reliability and operational parameters. In this paper we discuss an approach in which a system with scheduled inspections and possible repair activities is described by the discrete state-transition deterioration model. The model can be translated into a semi-Markov process which, after solving, yields numerous reliability characteristics including average equipment life, its deterioration rate represented by so called life curve, probability of failure within given time horizon, etc. The analysis is performed for particular maintenance policy which has been integrated in the model and by adjusting the model to specific changes in this policy (e.g., modifications of repair frequencies) their consequences for system reliability can be evaluated. In the text we present briefly the methodology behind model creation and concentrate on its one specific aspect: automatic adaptation of the model to the adjusted maintenance policy with modified frequencies of repairs. In particular, it is shown that such adaptation can be reduced to the adjustment of transition probabilities that are found in the model, and such adjustment can be numerically implemented with one of the standard root-finding algorithms. Upon presentation of the adjustment method, its use is illustrated by a description of the computer tool which helps in evaluating reliability and financial effects of changes in maintenance policy of an ageing equipment.

Keywords: *state-transition deterioration model, semi-Markov process, reliability analysis, root-finding algorithm, maintenance analysis*

1. Introduction

Efficient and, at the same time, cost-effective maintenance is an important element of reliable operation in contemporary complex technical systems. Selecting the optimal maintenance strategy must take numerous issues into account and among them reliability and economic factors are often of equal importance. On one side, it is obvious that for successful system operation failures must be avoided and this opts for extensive and frequent maintenance activities. On the other, superfluous maintenance may result in very large and unnecessary cost. Finding a reasonable balance between these two is a key point in reliable system operation.

In order to be able to plan such maintenance appropriate models are necessary that would represent equipment deterioration process and, at the same time, would take into account various maintenance operations. In this paper we present a methodology that assists a person who decides about maintenance activities by evaluating risks and costs associated with choosing different maintenance strategies. Instead of searching for a globally optimal solution to a problem: "what maintenance strategy would lead to the best reliability and dependability parameters of system operation", in this approach different maintenance scenarios can be examined in "what-if" studies and their reliability and economic effects can be compared so that a person managing the maintenance is assisted in making informed decisions ([1–3]).

The method has been presented initially in [1] and its specific extensions were further described in [4–8]. In this work, we summarize the current state of development and concentrate on one important aspect of the methodology: fully automatic adjustment of the model to possible modifications of the maintenance policy which are often required for studies requested by the user. Also we will provide an illustration of the application of the method presenting a computer tool which helps in evaluating reliability and financial effects of changes in maintenance policy of an ageing equipment.

The main contents of the paper is divided into three parts. The first one (section 2) presents general description of the methodology which is based on state-transition deterioration models, semi-Markov processes and the concept of a life curve used for visualisation of equipment ageing. The problem of automatic model adjustment is presented in the second part (section 3) which includes detailed discussion of possible numerical algorithms that can be implemented for probability approximation. Finally, in the last part (section 4) the method of model adjustment is used in a computer tool which uses the concept of a life curve and discounted cost to study the effect of the equipment ageing under different hypothetical maintenance strategies

2. Modelling the Ageing Process

There are numerous factors that have an effect on the ageing process of equipment that undergoes scheduled inspections and maintenance actions. Among them there are various aspects related to its physical characteristics, operating practices, and the maintenance policy. The method described in this paper uses a generic model that assumes that the equipment will deteriorate in time and, if not maintained, will eventually fail. To counteract, the scheduled inspections are performed and if the deterioration process is discovered, preventive maintenance is applied which can restore the condition of the equipment. Such a maintenance activity will return the system to a specific state of deterioration, whereas repair after failure will restore to "as-new" condition ([9–10]). With these assumptions, the maintenance policy components that must be recognized in the model are: monitoring or inspection (how the equipment state is determined), the decision process (which determines the outcome of the decision), and, finally, the maintenance actions – the repairs (or possible decision outcomes).

2.1. The State-Transition Model

One of the approaches that can properly incorporate all the above suppositions about the aging process and maintenance activities is based on state-space (Markov) model ([11–16]). The model consists of the states the equipment can assume in the process, and the possible transitions between them.

The method described in this paper uses a model of the Asset Maintenance Planner (AMP) that was initially developed and implemented by George J. Anders and Henryk Maciejewski ([17–18]). The AMP model is designed for equipment exposed to deterioration but undergoing maintenance at prescribed times. It computes the probabilities, frequencies and mean durations of the states of such equipment. The basic ideas in this approach are the probabilistic representation of the deterioration process through discrete stages, and the provision of a link between deterioration and maintenance.

For structure of a typical AMP model see Figure 1. In the model, the deterioration progress is represented by a chain of deterioration states $D1 \ldots DK$ which then leads to the failure state F. In most situations, it is sufficient to represent deterioration by three stages: an initial ($D1$), a minor ($D2$), and a major ($D3$) stage of deterioration ($K = 3$). This last is followed, in due time, by equipment failure (F) which requires extensive repair or replacement.

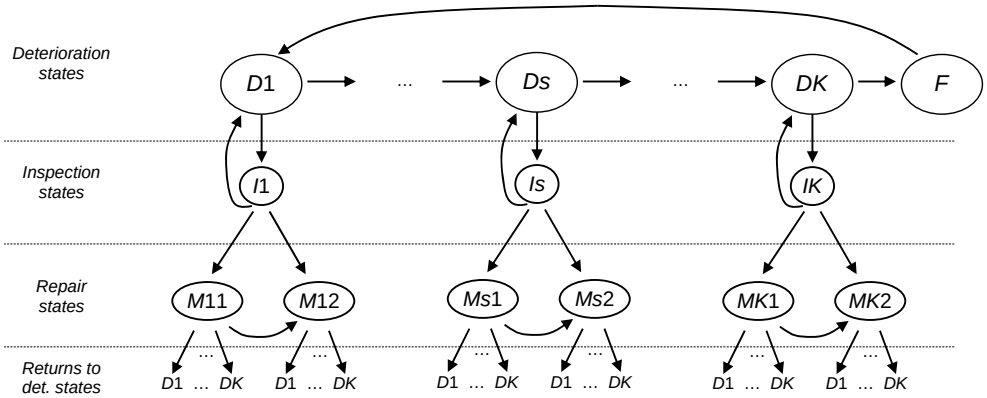

Figure 1. The state-transition model representing the deterioration chain with inspection and repair states (an example with two types of repairs is shown)

In order to slow deterioration and thereby extend equipment lifetime, the operator will carry out maintenance according to some pre-defined policy. In the model of Figure 1, regular inspections (I*s* states) are performed which result in decisions to continue with, e.g., minor ($Ms1$) or major ($Ms2$) repair (more than two types of repairs can be modelled), or to return to the deterioration state Ds without any repair. The expected result of all maintenance activities is a single-step improvement in the deterioration chain; however, it is possible to take into account also cases where no improvement is acquired or even where some damage is done through human error in carrying out the maintenance, which results in returning to the stage of more advanced deterioration.

The choice probabilities (at transitions from inspection states) and the probabilities associated with the various possible outcomes are based on user input and can be estimated, e.g., from historical records or operator expertise. Therefore, creation of the model and then its fine-tuning to some real historical data of equipment operation and maintenance records is a complex task that requires expert intervention.

Mathematically, the state-transition model of Figure 1 can be translated into a semi-Markov process, and then solved by the well-known procedures. The solution will yield all the state probabilities, frequencies and mean durations. Another technique, employed for computing the so-called first passage times (FPT) between states, will provide the average times for first reaching any state from any other state. If the end state of the passage is F, the FPTs are the mean remaining lifetimes from any of the initiating states. For state $D1$ this estimates the expected equipment life for the maintenance policy that has been incorporated in the model.

2.2. Using the Model in Reliability Analysis

A convenient way to represent the deterioration process is by the life curve of the equipment ([9]). Such a curve (see the first graph in Figure 2) shows the relationship between asset condition, expressed in either engineering or financial terms, and time. This concept is easy to comprehend for a non-expert end user (for example, a manager analysing various options of the maintenance policies) who does not need to know all the intrinsic details of state-transition models and Markov processes.

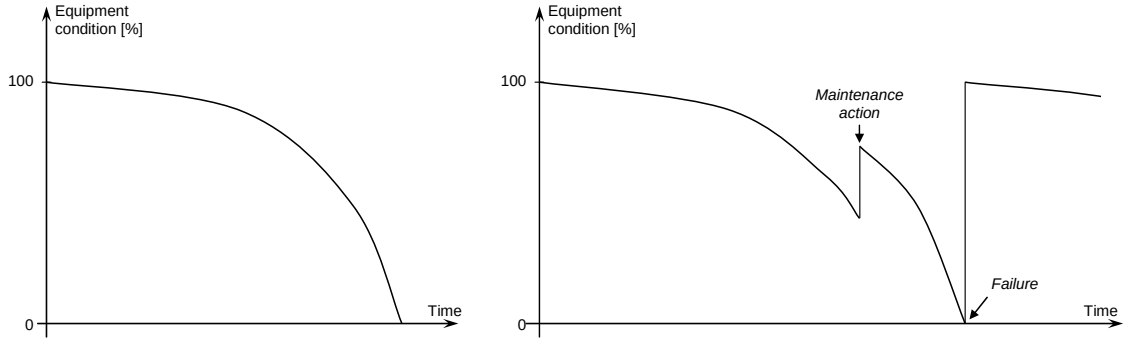

Figure 2. A simple life curve computed from the semi-Markov model (left) and a complex life curve representing some maintenance scenario (right)

A life curve that corresponds to some given Markov model can be created as follows. As pointed out above, computing the average first passage time (FPT) from the first deterioration state (D1) to the failure state (F) yields an average lifetime of the equipment, i.e., the length of the curve. On the other hand, solving the model for the state probabilities makes possible computing the expected state durations, which are then used to determine the shape of the curve, i.e. the rate of deterioration over different phases of equipment wear (some additional decisions are required as to how the deterioration states are mapped to ranges of the asset condition values).

Simple life curves obtained for specific maintenance policies (i.e. specific models) can later be combined in composite life curves which describe possible complex maintenance scenarios ([1], [8]). The second graph in Figure 2 shows an example of such a scenario when a preventive maintenance is performed at some moment in time (restoring the asset condition to approx. 80%) after which the failure occurs leading to equipment replacement. This curve is composed of three segments of simple life curves and they do not need to represent the same models, i.e. a situation can be modelled when a change in maintenance policy takes place, e.g. after the repair.

Furthermore, having the model and its (simple) life curve, one can compute the probability of failure (PoF) within given time period T for the equipment which currently is in some specific deterioration condition AC. The procedure is as follows:

(1) for the current value of asset condition AC, find from the life curve the corresponding deterioration state Dc and then compute a state progress SP (%), i.e. estimate how long the equipment has already been in the Dc state

(2) running FPT analysis on the model, find the probability distribution functions $d_c(t)$ and $d_{c+1}(t)$ of First Passage Time from the current state Dc and the subsequent deterioration state $D(c + 1)$, to the failure state F

(3) interpreting the state progress SP as a weight which balances the current equipment condition between Dc and $D(c + 1)$, estimate the final value of the probability as:

$$\mathrm{PoF}(T) = d_c(T) \cdot (1 - \mathrm{SP}) + d_{c+1}(T) \cdot SP. \quad (1)$$

When equipment deterioration over the time period T is represented by a complex life curve, then the above procedure must be applied to the every simple curve segment which is included in T.

3. Adjusting the Model to Different Repair Frequencies

In practical applications when different maintenance policies need to be investigated, one of the most important aspect of system operation is appropriate representation of various types and frequencies of repairs that may be included in possible maintenance policies under consideration. In the approach presented in this paper it is the task of the expert to properly incorporate these characteristics of the original ("continue as before") policy in the model, but once such model is available the non-expert end user may wish to analyse various hypothetical policies that are created by simple modifications of the original repair frequencies. For example, the end user may wish to consider an option "what if the minor repair rate is reduced by half" or "what if the minor and major repairs are removed completely but the medium repair is performed twice as often". Such studies require a mechanism that, having the expert-created model representing some original repair policy, would be able to generate a derived model that would correspond to the same equipment (i.e. the same deterioration chain as presented in Figure 1) but submitted to a modified repair policy with different repair frequencies. It should be noted that the same mechanism would be also helpful during construction of the model by the expert when fine-tuning of repair frequencies is needed in order to achieve compliance of the model with some real-world repair policy stored in historical records of equipment operation.

3.1. The Adjustment Procedure

Let's assume that the deterioration model under consideration consists of K deterioration states and R repairs. Also, let P^{sr} denotes probability of selecting maintenance r in state s (assigned to the decision after inspection state Is) and P^{s0} represents probability of returning to state Ds from inspection Is which corresponds to a situation when no maintenance is scheduled as a result of the inspection. The foremost condition that must be met at all times is that in all deterioration states $s = 1 \ldots K$:

$$\mathrm{P}^{s0} + \sum_r \mathrm{P}^{sr} = 1. \quad (2)$$

Let F^r represents the frequency of some repair r as it is generated by the model. The problem of model adjustment can be formulated with various assumptions and with different goals in mind but in this approach it is defined as follows:

> Given an initial semi-Markov model M_0, with internal structure representing deterioration, inspection and repair states as described above and producing the initial vector of repair frequencies $\mathbf{F}_0 = [\mathrm{F}_0^1, \mathrm{F}_0^2 \ldots \mathrm{F}_0^R]$, modify the probabilities P^{sr} assigned to transitions from inspections states Is so that the resulting model generates some requested vector of goal frequencies $\mathbf{F}_\mathrm{G}$.

Usually, the vector $\mathbf{F}_\mathrm{G}$ may represent the observed historical values of the repair frequencies (when an expert user is working on fine-tuning of the initial model) or some hypothetical frequencies (when an end user investigates possible modifications of the repair policy).

There are numerous approaches that can be used in order to accomplish such model adjustment. In the proposed solution, an iterative approximation approach has been chosen in order to preserve an original construction of the model M_0 as mush as possible. In this method a sequence of tuned models $M_0, M_1, M_2, \ldots M_N$ is evaluated in N steps with each consecutive model approximating desired goal with a better accuracy. Starting with $i = 0$, the procedure consists in the following steps:

1° for the current model M_i compute its vector of repair frequencies $\mathbf{F}_i$

2° evaluate an error of M_i as a distance between vectors $\mathbf{F}_\mathrm{G}$ and $\mathbf{F}_i$

3° if the error is within the user-defined limit ε, consider M_i as the final model and stop the procedure ($N = i$); otherwise proceed to the next step

4° create a new model M_{i+1} by *tuning* values of P_i^{sr}, then correct P_i^{s0} according to condition (2)

5° return to the step 1° for the next iteration.

The error computed in step 2° can be expressed in may ways. As the absolute values of repair frequencies may vary in a broad range within one vector $\mathbf{F}_i$, yet the values of all are significant for model evaluation, the relative measures work best in practice:

$$\left\|\mathbf{F}_{\mathrm{G}}-\mathbf{F}_i\right\|=\frac{1}{R}\sum_{r=1}^{R}\left|\mathrm{F}_i^r/\mathrm{F}_{\mathrm{G}}^r-1\right| \tag{3}$$

or

$$\left\|\mathbf{F}_{\mathrm{G}}-\mathbf{F}_i\right\|=\max_r\left|\mathrm{F}_i^r/\mathrm{F}_{\mathrm{G}}^r-1\right|. \tag{4}$$

The latter formula is more restrictive: it ensures that any repair does not differ from the goal more than the imposed limit and therefore this version was used in the numerical implementation of the method that is considered in this paper.

3.2. Generation of the Adjusted Model

Of all the steps that compose the iteration run, it is clear that adjusting probabilities P_i^{sr} in step 4° is the heart of the whole method. This is accomplished with the following two assumptions that are introduced not only to simplify the task, but also to improve quality of the result.

The first assumption is related to the fact that although, in general, the P^{sr} probabilities represent $K{\cdot}R$ free parameters that could be freely modified in order to arrive at the requested goal, their uncontrolled modification can lead to serious deformation of the model and this should be avoided. To this point a restrictive condition is adopted: if the probability of some particular repair must be modified, it is modified *proportionally* in *all* deterioration states, so that during the adjustment the proportion between this repair probabilities over all states remains unchanged and are the same as in the initial model M_0:

$$\forall\ i, r\quad \mathrm{P}_0^{1r}:\mathrm{P}_0^{2r}:\ldots:\mathrm{P}_0^{Kr}\ \sim\ \mathrm{P}_i^{1r}:\mathrm{P}_i^{2r}:\ldots:\mathrm{P}_i^{Kr}. \tag{5}$$

This assumption also significantly reduces dimensionality of the problem, because now only R scaling factors, denoted as the vector $\mathbf{X}_{i+1}=[\,\mathrm{X}_{i+1}^1,\ \mathrm{X}_{i+1}^2,\ \ldots\ \mathrm{X}_{i+1}^R\,]$, must be found to compute all new probabilities required to create the model M_{i+1}:

$$\mathrm{P}_{i+1}^{sr}=\mathrm{X}_{i+1}^r\cdot\mathrm{P}_0^{sr},\quad r=1\ldots R,\ s=1\ldots K. \tag{6}$$

Moreover, and this observation leads to the second assumption, although the frequency of a repair r depends on the probabilities of all repairs (modifying probability of one repair changes, among others, state durations in the whole model; thus, it changes the frequency of all states) it can be assumed that, in a case of a single-step small adjustment, its dependence on repairs other than r can be considered negligible and

$$\mathrm{F}_i^r=\mathrm{F}_i^r\left(\mathrm{X}_i^1,\mathrm{X}_i^2\ldots\mathrm{X}_i^R\right)\approx\mathrm{F}_i^r\left(\mathrm{X}_i^r\right). \tag{7}$$

With these two assumptions, generation of a new model in step 4° of the above described procedure is reduced to the problem of solving R non-linear equations in the form of

$$\mathrm{F}_i^r\left(\mathrm{X}_i^r\right)=\mathrm{F}_G^r\ . \tag{8}$$

This task can be accomplished with one of the standard root-finding algorithms which will be presented in the next point.

One aspect of the procedure requires additional attention, though: applying equation (6) with $X_{i+1} > 1$ may violate condition

$$\sum_r\mathrm{P}_{i+1}^{sr}\le 1 \tag{9}$$

in some deterioration state s. This situation needs special tests that would detect such illegal probability values and then reduce them proportionally so that their sum does not exceed 1: a so called *scale-down transformation* needs to be applied. As the case studies show such situations do occur during model tuning towards repair frequencies that are remarkably higher than in the initial model M_0. In its simplest

form, the scale-down operation consists in dividing each probability P^{sr} in the offending state *s* by the sum of all repair probabilities in this state, as they are computed with equation (6) without scaling:

$$P^{sr} = P^{sr} \big/ S_{Ds}, \quad S_{Ds} = \sum_{r=1}^{R} P^{sr}. \tag{10}$$

This will also lead to $P^{s0} = 0$ which means that every inspection ends with some repair and there are no direct returns from inspection state *Is* to deterioration state *Ds*. Moreover, this obligatory correction mechanism can result in violation of the proportionality rule (5) as an unavoidable side effect. In such cases modification of the model is more serious than the initial assumptions allow but this must be tolerated if the goal repair frequencies requested by the end user are to be achieved.

3.3. Numerical Methods Used for Probability Estimation

With all the assumptions regarding scaling factors and their influence on repair frequencies, generation of a new model is now reduced to the problem of solving *R* equations in the form of (8). This can be accomplished with one of the standard numerical algorithms for finding roots of a non-linear function. The method described in this paper has been tested with implementation of the following three algorithms: the Newton method working on linear approximation of $F_i^r()$, the secant method and the false position (*falsi*) method.

3.3.1. Newton method on Linear Approximation (NOLA)

In this solution it is assumed that $F_i^r()$ is a linear function defined by points $F_i^r(X_i)$ (obtained after solving the model in step 1°) and $F_i^r(0)$ (which can be assumed to be equal zero). Then simply

$$X_{i+1}^r = F_G^r / F_i^r. \tag{11}$$

Noteworthy advantage of this approach lies in the fact that no other solution than the current frequency $F_i^r(X_i)$ is required to compute the next approximation, so errors of previous steps do not accumulate and convergence is good from the first iteration (the method has no memory effects).

Table 1 and Figure 3 present the details of exemplary model adjustment with this method implemented. The sample model consisted of three deterioration states and three types of repairs: minor, medium and major ($K = R = 3$). The values of goal frequencies has been selected to be ≈ 50% of that in the original repair policy ($\mathbf{F}_G = ½\mathbf{F}_0$) which corresponds to some hypothetical repair policy "what if repair frequencies of this piece of equipment are reduced by half".

Table 1. Sample model adjusted to $\mathbf{F}_G = ½\mathbf{F}_0$ with the NOLA method, $\varepsilon = 1E\text{-}4$

i	Relative freq. of repairs F_i^r / F_G^r			Error	Scaling factors X_{i+1}^r		
	***r* = 1**	**2**	**3**		***r* = 1**	**2**	**3**
0	2.007762	2.000581	2.049800	1.049814	0.49810	0.49990	0.48780
1	1.127838	1.230000	1.254300	0.254331	0.88670	0.81300	0.79720
2	1.018676	1.039839	1.044400	0.044398	0.98170	0.96170	0.95750
3	1.002895	1.006516	1.007300	0.007255	0.99710	0.99350	0.99280
4	1.000467	1.001065	1.001200	0.001179	0.99950	0.99900	0.99880
5	1.000076	1.000161	1.000200	0.000183	0.99990	0.99980	0.99980
6	1.000010	1.000032	1.000000	0.000040	1.00000	1.00000	1.00000

As it can be seen from the submitted data, the adjustment process goes smoothly and without any perturbations: the average convergence ratio is nearly constant and the model firmly approaches the required goals for all three frequencies. As the goal vector has been assumed to be 50% of the frequencies in the original (initial) model, the relative error in the first iteration equals to approx. 100% (the frequencies are twice as large as required) and in each subsequent iteration it is reduced by a factor of 4 to even 6. Finally, the imposed accuracy of 1E-4 (0.01%) is reached after 6 steps. It should be noted that such high precision in model tuning has been selected for illustrative purposes in this paper, while in practical engineering cases accuracy of 1% is more than adequate; in the discussed example such level is obtained after just three steps.

3.3.2. The Secant Method

In this standard technique the function is approximated by the secant defined by the last two approximations computed for points X_{i-1}^r, X_i^r; thus the new solution is calculated as:

$$X_{i+1}^r = X_i^r - \frac{X_i^r - X_{i-1}^r}{F_i^r - F_{i-1}^r}\left(F_i^r - F_G^r\right) \cdot \tag{12}$$

After that X_{i-1}^r is discarded and X_{i+1}^r and X_i^r are considered as the pair defining the secant for the next iteration.

To begin the procedure two initial points are needed. In this approach we propose to choose the first point equal to the initial frequency of the model M_0 ($X_0^r = 1$), while the second point is computed as in the NOLA method: $X_1^r = F_G^r / F_0^r$. Having the pairs X_0^r, X_1^r ($r = 1, 2, \ldots R$) the procedure starts according to equation (12) in iteration $i = 1$ and continues so in further steps.

In our exemplary case this method produced not-so constant convergence rate compared to the NOLA method presented in the previous point. Looking at the error values listed in table 2 for every iteration, there are some steps with very good improvement (e.g. error reduction from -0.1 to 0.008 for $i = 3$) but those are followed by mediocre progress in the next iteration (reduction from 0.008 to 0.006 for $i = 4$). Nevertheless, this does not exclude the method from application in the discussed system: the required accuracy is achieved in just one additional step ($i = 7$ compared to 6 for the NOLA method) and this is still an acceptable result.

Table 2. The same adjustment case ($\mathbf{F}_G = ½\mathbf{F}_0$, $\varepsilon = 1E\text{-}4$) controlled by the secant method

i	**Relative freq. of repairs** F_i^r / F_G^r			**Error**	**Scaling factors** X_{i+1}^r		
	***r* = 1**	**2**	**3**		***r* = 1**	**2**	**3**
0	2.007762	2.000581	2.049800	1.049814	0.49810	0.49990	0.48780
1	1.127838	1.230000	1.254300	0.254331	0.85360	0.70140	0.66440
2	0.990467	0.918355	0.893500	-0.106504	1.01190	1.11150	1.14910
3	0.995514	1.004032	1.007800	0.007832	1.01050	0.99530	0.99110
4	1.006038	0.999935	0.999600	0.006037	0.99410	1.00010	1.00040
5	1.000219	1.000161	1.000200	0.000218	0.99980	0.99990	0.99990
6	1.000000	1.000129	1.000100	0.000120	1.00000	0.99980	0.99990
7	1.000010	0.999903	1.000000	-0.000084	1.00000	1.00010	1.00000

The graphs in Figure 3 provide an additional explanation of the problem for this particular case. It can be seen that for $i = 3$ the method proposed actually a very good approximation but with just a little overshoot and this caused problems in the next iteration. This is a known setback of the secant method which is removed in the *falsi* method.

3.3.3. The False Position (Falsi) Method

In this approach X_{i+1}^r is computed as in (12) but the difference lies in choosing points for the next iteration. While in the secant method always X_{i-1}^r is dropped, now X_{i+1}^r is paired with that one of X_i^r or X_{i-1}^r which lies on the opposite side of the root. In this way when (12) is applied the solution is bracketed between X_i^r and X_{i-1}^r (which is the essence of the *falsi* method).

As in 3.3.2, the two initial points are needed but now they must lie on both sides of the root, i.e.

$$\left(F_0^r - F_G^r\right) \cdot \left(F_1^r - F_G^r\right) < 0 \tag{13}$$

Choosing such points may pose some difficulty. To avoid multiple sampling, as in the secant method it is proposed to select $X_0^r = 1$ and to compute X_1^r like in NOLA method, but now with some "overshoot" that would guarantee (13):

$$X_1^r = \left(F_G^r / F_0^r\right)^{\alpha} \tag{14}$$

with a new parameter $\alpha > 1$ controlling the overshoot effect. The overshot must be sufficient to ensure condition (13) but, on the other hand, it should not produce too much of an error as this would deteriorate approximation process during initial steps and would produce extra iterations. If (13) is not met by initial value of X_1^r (14) can be re-applied with an increased value of α, although it should be noted that each

such correction requires solving a new M_1 model and in effect this is the extra cost almost equal to that of the whole iteration.

Table 3. The sample case ($\mathbf{F}_G = ½\mathbf{F}_0$, $\varepsilon = 1E\text{-}4$) adjusted by the *falsi* method

i	Relative freq. of repairs F_i^r / F_G^r			Error	Scaling factors X_{i+1}^r		
	***r* = 1**	**2**	**3**		***r* = 1**	**2**	**3**
0	2.007762	2.000581	2.049800	1.049814	0.24810	0.24990	0.23800
1	0.591533	0.686129	0.691200	-0.408464	1.87420	1.71690	1.72780
2	1.067305	1.086871	1.092200	0.092239	0.93400	0.90950	0.90310
3	1.004705	1.005032	1.004800	0.005045	0.99510	0.99430	0.99450
4	1.000238	1.000258	1.000300	0.000313	0.99980	0.99970	0.99960
5	1.000010	1.000000	1.000000	0.000014	1.00000	1.00000	1.00000

Theoretical characteristics of the *falsi* technique are well illustrated by the data shown in Table 3 and Figure 3. Of all three methods presented here, this one generated the requested accuracy with the minimum number of iterations: 6 vs. 7 of NOLA and 8 of secant. The α parameter that controls the overshoot effect in the first approximation was equal 2 and this indeed generated an overestimation in the model M_1 but this was properly and quickly compensated in the steps that followed. As a result, the convergence flow which can be seen in Figure 3 is close to the ideal case with two deviations from the target: an overestimation immediately followed by an underestimation after which the solution arrives at the final goal.

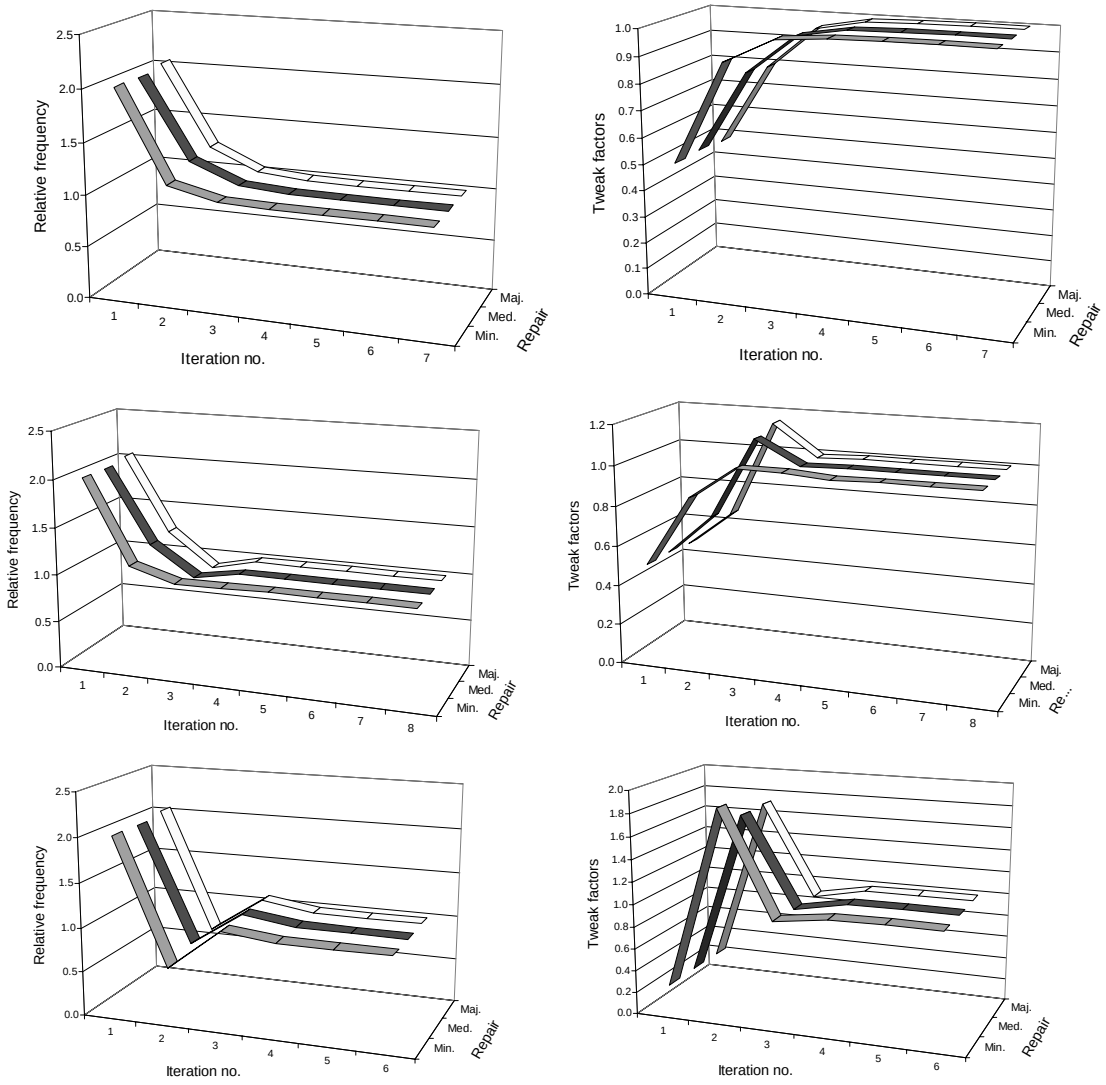

Figure 3. Adjusting the sample model to $\mathbf{F}_G = ½\mathbf{F}_0$ ($\varepsilon = 1E\text{-}4$) with three different numerical methods: NOLA (top), secant (middle) and *falsi* (bottom)

3.3.4. Evaluation of the Methods

Figure 4 illustrates efficiency of the three approximation methods in another typical adjustment cases. For these tests the model used in the previous examples was adjusted to the policies $\mathbf{F}_G = [0, 0, F_0^3]$ (only major repair is performed with the other two types removed) and $\mathbf{F}_G = ¼\mathbf{F}_0$ (frequencies of all repairs reduced to 25%). A short comment should be made about the first of these examples. With $\mathbf{F}_G = [0, 0, F_0^3]$, it is very easy to remove any repair from the model by simply assigning $P^{sr} = 0$ in all the states, effectively increasing the value of P^{s0} at the same time. In such case no adjustment is required as the requested goal ($F^r = 0$) is achieved immediately. Nevertheless, such modification does affect the frequencies of other repairs that are left in the model (F^3 in this case) and the adjustment mechanism is still necessary just to return them to their requested values.

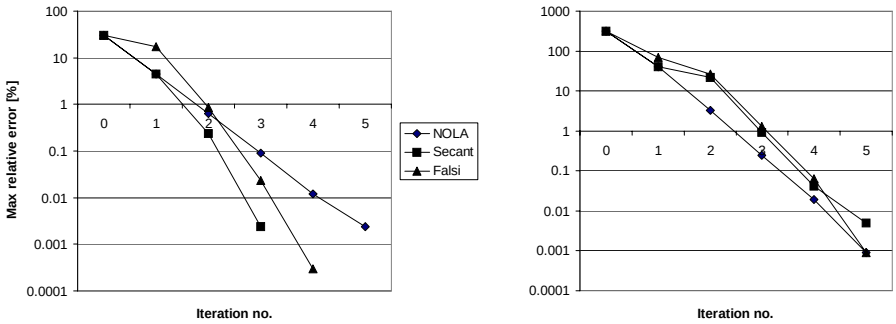

Figure 4. Convergence rate of the three approximation methods for two sample maintenance policies: $\mathbf{F}_G = [0, 0, F_0^3]$ (left) and $\mathbf{F}_G = ¼\mathbf{F}_0$ (right); $\varepsilon = 0.01\%$ in both cases

Comparing the effectiveness of the methods it should be noted that although simplifications of the NOLA solution may seem critical, in practice it works quite well. As it was noted before, due to its simplicity this method has one advantage over its more sophisticated rivals: since computation of the next solution does not depend on previous approximations, selection of the starting point is not so important and the accuracy during the first iterations is often better than in the secant or *falsi* cases. For example, in the case used in 3.3.1 ÷ 3.3.3 the NOLA method reached accuracy of 4.4% already after 2 iterations, while for secant and falsi methods the error after two iterations was, respectively, 11% and 9.2%. Superiority of the latter methods, especially of the *falsi* algorithm, manifests itself in the later stages of the process when the potential problems with an initial selection of the starting points have been diminished.

4. Asset Risk Manager

In this section we will present the "Asset Risk Manager" (ARM) software package which uses the concept of a life curve and discounted cost to study the effect of equipment ageing under different hypothetical maintenance strategies [1]. The results generated by the program are based on Markov models that were presented in the previous sections and in many cases the models were adjusted to the modified maintenance policies as it has been described in section 3.

For the program to generate automatically the life curves, default Markov model for the equipment has to be built and stored in the computer database. This is done through the prior running of the AMP program by an expert user. Therefore, both AMP and ARM programs are closely related, and usually, should be run consecutively.

Implementation details of Markov models, tuning its parameters and all other internal particulars should not be visible to the non-expert end user. All final results are visualized either through an easy-to-comprehend idea of a life curve or through other well-known concepts of financial analysis. Still, prior to running the analysis some expert involvement is needed, largely in preparation, importing and adjusting AMP models.

4.1. User input

A typical study is described through a comprehensive set of parameters that are supplied by a non-expert end user. They fall into three broad categories.

(A) General data. The Markov model of the equipment in question and its current state of deterioration form the primary information that is the starting point to most of ARM computations. The Markov model represents the equipment with present maintenance policy and is selected from a database of imported AMP models which need to be prepared by an expert in advance. Deterioration state, referred to as "Asset Condition" (AC) throughout the ARM, must be supplied by the end user as a percentage of "as-new" condition. Besides, a number of additional general parameters need to be specified, such as the time horizon over which the analysis will be performed, discount and inflation rates for financial calculations etc.

(B) Description of the present maintenance policy. It is assumed that three types of maintenance repairs can be performed: minor, medium and major. These correspond to the appropriate states in Markov model and not all of them must be actually present in the policy. For each repair user supplies its basic attributes, e.g. cost, duration and frequency.

(C) List of alternative actions. These are the hypothetical maintenance policies that decision-maker can chose from. Each action is defined as one of four types:

- continue as before (i.e. do not change the present policy),
- do nothing (i.e. stop all the repairs),
- refurbish,
- replace.

Apart from the first type, every action can be delayed for a defined amount of time. Additionally, for "non-empty" actions (i.e. any of the last two types) user must specify what to do in the period after action; the choices are:

(a) to change type of equipment and / or
(b) to change maintenance policy.

In case of option (b), the new maintenance policy will require a different Markov model which, in turn, may need to be generated by the adjustment algorithm discussed in section 3. Such generation is done automatically on the background of the system operation and the end user does not even know about this fact.

For every action the user must also specify what to do in case of failure: whether to repair or replace the failed equipment, its condition afterwards, cost of this operation etc. Thank to these options a broad range of maintenance situations can be described and then analysed.

The first action in the list is always "Continue as before" and this is the base of reference for all the others. The ARM can be directed to compute life curves, cost curves, or probabilities of failure – for each action independently – and then to visualize computed data in many graphical forms to assist the decision-maker in effective action assessment.

It should be noted that while the need for some action (e.g., overhaul or change in maintenance policy) is identified at the present moment, the actual implementation will usually take place only after a certain delay during which the original maintenance policy is in effect. Using ARM it is possible to analyze the effect of that delay on the cost and reliability parameters.

4.2. Life Curves

As it has been pointed out before, computing the average first passage time (FPT) from the first deterioration state (D1) to the failure state (F) in the Markov model yields the average lifetime of the equipment, i.e. length of its life curve. On the other hand, solving the model for the state probabilities of all consecutive deterioration states makes possible computing the state durations, which in turns determine the shape of the curve. Simple life curves obtained for different maintenance policies are later combined in constructing composite life curves which describe various maintenance scenarios.

For sake of simplicity and consistency, exactly three deterioration states, or levels, are always presented to the end user: minor, medium and major, with adjustable AC ranges. In case of Markov models which have more than three D*s* states, the expert decides how to assign the Markov states to three levels when importing the model.

Figure 5 shows exemplary life curves computed by ARM for typical maintenance situations. In each case the action is delayed for 3 time units (months, for example) and the analysis is performed for a time horizon of 10 time units. In case of failure seen in "Do nothing" action, equipment is repaired and its condition is restored to 85%.

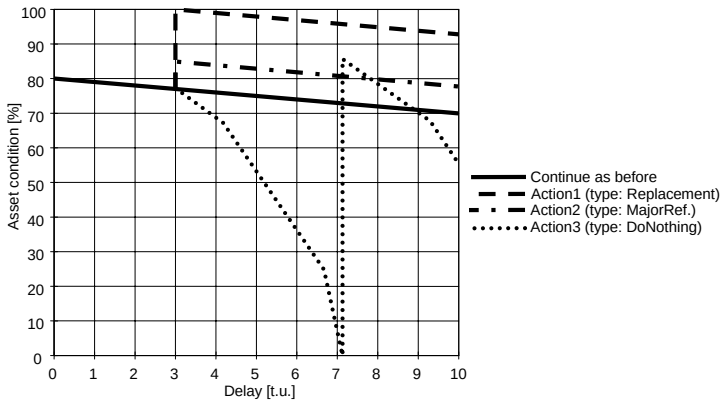

Figure 5. Life curves computed for three different actions ("Action1" … "Action3") and compared to the present maintenance policy ("Continue as before")

4.3. Probability of Failure

For a specific action, probability of failure within the time horizon (PoF_{TH}) is a sum of two probabilities: of failure taking place before (PoF_B) and after (PoF_A) the moment of action. It is assumed that failures in these two periods making up the time horizon are independent, so

$$PoF_{TH} = PoF_B + PoF_A - PoF_B \cdot PoF_A . \tag{15}$$

To compute $PoF(T)$ within some time period T, the Markov model for the equipment and the life curve are required. Then, the procedure for computation of PoF_A and PoF_B can be applied as it has been described in section 2.2, equation 1.

For better visualization, rather than finding a single PoF_{TH} value for action defined by the user in input parameters, ARM computes a curve which shows the PoF_{TH} as a function of action delay varying in a range 0 ÷ 200% of user-specified initial value. An example is demonstrated in Figure 6 for "Do nothing" action (user-defined delay = 3 time units), where also the two probability components PoF_B and PoF_A are shown.

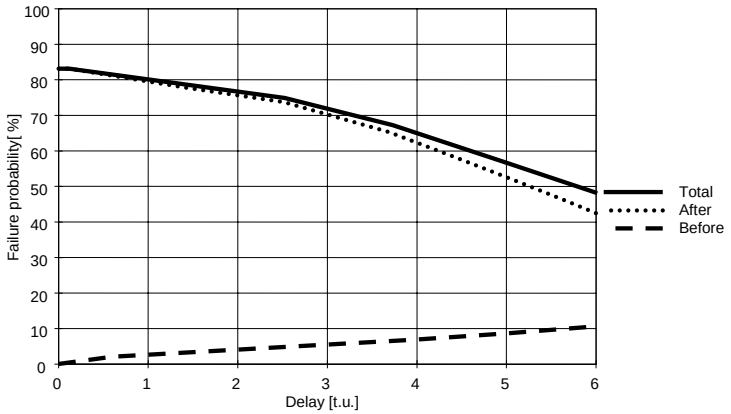

Figure 6. Probability of equipment failure within the time horizon for a modelled action, computed as a function of action delay

A more detailed analysis is shown in Figure 7. In this case we consider a situation when, with initial equipment deterioration estimated as 80% of "as-new" condition, some specific actions – a repair or just a change in maintenance policy – will take place after a 3-year delay while the effects will be evaluated for a 10-year time period. The actions in the scenarios will be as follows:

- adopting "do nothing" policy, which means just stopping all inspections and repairs; in case of failure the equipment will be repaired and its condition restored to 85%,
- replacing the equipment with "as-new" one and then switching to the "do nothing" policy,
- performing a major refurbishment of the equipment which restores its condition to 85% and then continuing with a medium repair only.

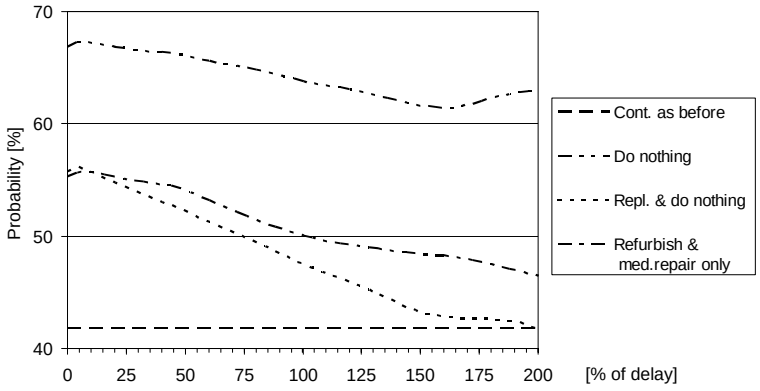

Figure 7. Probability of equipment failure within a period of 10 years for different cases of maintenance scenarios

Probability of failure within the time horizon computed for these strategies is shown in Figure 7. Values on the graphs are presented as functions of the action delay time (100% = 3 years) and they are compared against the probability of failure for the unmodified standard maintenance ("continue as before"). The value of this probability has been computed to be 42%.

It can be seen in case of all three scenarios that, since the new maintenance policy after the action is more or less reduced, the more the action is delayed, the less probable equipment failure becomes. For evident reasons adopting "do nothing" policy leads to the highest values of the failure probability, while replacing the equipment and "doing nothing" afterwards turned out to be a less dangerous strategy (in terms of failure probability) than refurbishing and then keeping only the medium repair. Whether the differences in the economic expenses of these two possible strategies justify this discrepancy in the reliability parameter or not – it remains an open question in further cost analysis and generally depends on the costs associated with the equipment failures.

One interesting observation can be made about the curve for "do nothing" strategy: its decrease is not strictly monotonic and there is a local minimum at the level of 61% for the delay equal to 164% (4.9 years) after which the probability begins to rise slowly. To explain this rise, two components: the probability of failure before and after the action should be investigated and they are shown in Figure 8. In general, these two components behave as expected: the later the action takes place, the higher the probability of failure before and the lower the probability of failure after the action but the rates of these two flows – increasing and decreasing – are not constant and do not sum up into a monotonic decrease. In this case, the probability of failure after the action falls down to some extent slower after the point of 164% and this causes the local minimum in the total probability of failure.

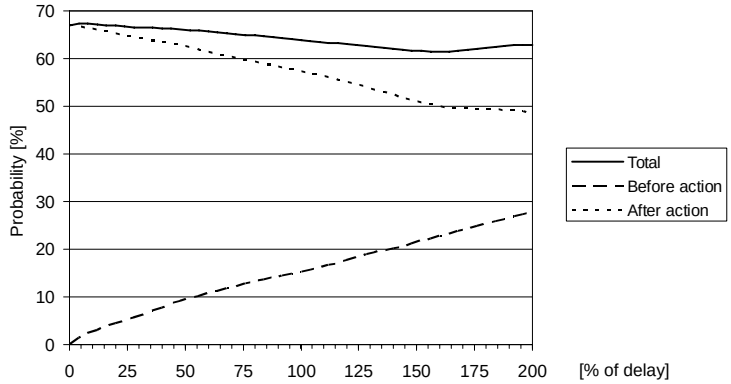

Figure 8. Probability of equipment failure before and after the action for "do nothing" scenario

4.4. Cost Curves

In financial evaluations the costs are expressed as present value (PV) quantities and this approach should also be used in this kind of studies because maintenance decisions on aging equipment include timing, and the time value of money is an important consideration in any decision analysis. The cost difference is often referred to as the Net Present Value (NPV). In the case of maintenance, the NPV can be obtained for several re-investment options which are compared to the "Continue as before" policy.
Cost evaluation for any maintenance scenario involves calculation of the following three fundamental classes of components:

1. cost of the maintenance activities,
2. cost of the selected action (i.e. refurbishment or replacement),
3. cost associated with failures (cost of repairs, system cost, penalties).

To compute the PV, inflation and discount rates are required for the specified time horizon. The cost of maintenance over the time horizon is the sum of the maintenance costs incurred by the original maintenance policy for the duration of the delay period (up to the action), and the costs incurred by the new policy for the remainder of the time horizon (after the action). The costs associated with the equipment failure over the time horizon can be computed similarly except that the failure costs before and after the action should be multiplied by the respective probabilities of failures, and two products added.

Figure 9 presents the plots showing the cost analysis for the scenario "replace and then do nothing". Again (as it has been in the case of probability of failure) the values are visualized as functions of the action delay varying in the range 0 ÷ 200% of user-specified reference value. The cost of replacement ("Action"), although does not depend on the delay, is not constant on the plot due to the PV calculations. It is also evident that delaying the action causes more repairs to be performed as elements of the present repair policy before "do nothing" becomes effective, hence several noticeable leaps appear in the maintenance cost flow.

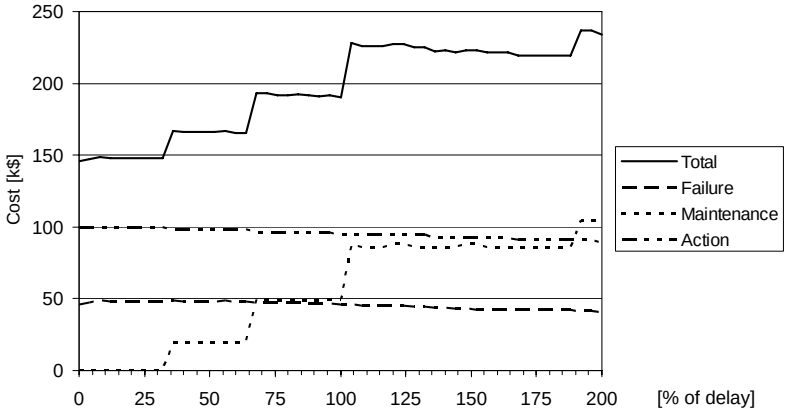

Figure 9. Estimated cost of "replace & do nothing" scenario (total value and the three components)

5. Conclusions

The paper described a modelling methodology which helps in choosing effective yet cost-efficient maintenance policy. Based on semi-Markov models representing the deterioration process, the equipment life curve and other reliability parameters can be evaluated. Once a database of equipment models is prepared, the end-user can perform various studies about different maintenance strategies and compare expected outcomes. As the results are visualized through the relatively simple concept of a life curve, no detailed expert knowledge about internal reliability parameters or configuration is required.
Additionally, the paper presented a method of model adaptation that allows automatic adjustment of the basic model to user-expected changes in maintenance policy. The numerical part of the method can be solved with common root-finding algorithms and it has proved its validity in numerous practical examples. Ability to adjust the models to such changes is crucial in practical studies of various possible maintenance scenarios.

Another important issue is how the adjustment modifies behaviour of the model in addition to reaching the desired repair frequencies and how the model should be constructed in order to accommodate the modifications without undesired side effects. For discussion of these problems please refer to [5–7]. In particular, while there is usually no problem with *reduction* of repair frequencies with the method described in this paper, special care must be taken when *increase* is requested because the probabilities cannot be enlarged indefinitely. In cases when the limit (2) is reached in all states even with $P^{s0} = 0$ and the requested frequencies are still not achieved, more substantial modifications of the model may be necessary. This also leads to specific directions as to how the model should be constructed in order to avoid such situations.

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HEURISTIC PREFERENCE RULES IN MULTIROUTE SCHEDULING

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A systematic account and comparative analysis of the efficiency of various statistical decision-making rules in scheduling theory, is undertaken. We will describe various adaptive randomisation preference rules in order to use the latter in industrial scheduling.

Keywords: *Heuristic preference rules; mn-permutations; SPT, LRT, FIFO and PC rules; Randomization preference rules*

1. Introduction

The area of our research centres on developing deterministic and statistical algorithms in job-shop scheduling, i.e., in determining quasi-optimal schedules for multiroute (job-shop) problems with n jobs and m machines [1–17].

The procedures described below for taking decisions in conflict situations (jobs queued up for a single machine) are, in essence, the simplest simulation models with which a quasi-optimal plan can be drawn up for feeding the jobs to the machine for a varied-route problem with n jobs and m machines. Thus, the paper under consideration is a further development of Refs [11–15]. We will henceforth use the terminology of [11].

Suppose that as a result of such a conflict situation at a certain moment of time t there is a queue of $r \le n$ jobs waiting to be fed into machine M. Denote the consecutive numbers of these jobs $q_1, q_2, ..., q_r$.

To devise a definite rule of preference means to work out such an algorithm as will make it possible to determine a numerical characteristic for each of the jobs in the queue $J_{q\xi}$, $\xi = \overline{1,r}$. The numerical characteristic is the preference coefficient

$$Q\left\{J_{q\xi}\right\} = \varphi\left\{T_t, T, t, q\xi\right\}, \tag{1}$$

where symbol T_t is the initial technological matrix T transformed by moment t (after excluding the operations already done by moment t). Job $J_{q\xi}$, which corresponds to optimal value $Q\left\{J_{q\xi}\right\}$, is selected from the queue and is fed first into the machine.

Depending on the random effect, the preference rules are subdivided into deterministic and random ones. The randomising procedure of preference rule Q centres on the following: first, the preference coefficients of the jobs in the queue $Q\left\{J_{q\xi}\right\}$, $\xi = \overline{1,r}$, are normalized by

$$Q^*\left\{J_{q\xi}\right\} = \frac{Q\left\{J_{q\xi}\right\}}{\sum_{\eta=1}^{r} Q\left\{J_{q\eta}\right\}}, \tag{2}$$

after which random value γ is simulated by the Monte Carlo method and equi-distributed in interval $[0;1]$. Job $J_{q\xi}$ is fed into the machine, there the relation exists

$$\sum_{\ell=1}^{\xi-1} Q^*\left\{J_{q\ell}\right\} < \gamma \le \sum_{\ell=1}^{\xi} Q\left\{J_{q\ell}\right\} \tag{3}$$

(sum $\sum_{1}^{0}$ is taken as equal to zero).

The physical idea of randomisation in preference rules consists in the fact that of the jobs demanding to be fed into the machine, the ones fed in usually have a larger preference coefficient. Note that if the use of deterministic preference rules always leads to the drawing up of a determined schedule (the assignment matrix can be determined by rule Q back in the stage preceding the start of the system's work by a preliminary simulation of the system's activity), in the case of randomisation principles, values S_{ij} are random in nature. As a result, in the deterministic version one can investigate the relative efficiency of two alternate preference rules Q_1 and Q_2 by comparing two determined values; the preference rule leading to the value of optimisation criterion K for a concrete technological matrix T is preferable. In a randomisation case, the preference of one rule or another should be decided by comparing statistical samplings (from the general totality of random values of criterion K), and applying the theory of checking statistical hypotheses.

We thus come to the statement of the basic shortcoming of preference rules: a comparative analysis of their efficiency can be made only for a concrete technological matrix T, and in turn, conclusions are to be drawn if T changes can also change. In fact, even if given a large number of existing preference rules proved in practice, one can always devise technological matrix T, for which we can draw up a new preference rule, more efficient than all the existing ones.

Let us now describe the best known and most widely applied preference rules.

2. Non-Combined (Local) Preference Rules

2.1. The Shortest Processing Time (SPT) Rule [16]

What this rule means is that when resolving a conflict situation, preferred is the job in set $Q\left\{J_{q\xi}\right\}$, $\xi = \overline{1,r}$, waiting to be fed into released machine M, which has the shortest operation duration. In other words,

$$Q\left\{J_{q\xi}\right\} = \frac{1}{t_{q\xi j\xi}}, \tag{4}$$

($t_{q\xi j\xi}$ being the duration of doing the $j\xi$-th operation of job $J_{q\xi}$ on machine M).

2.2. The Longest Remaining Time (LRT) Rule [16]

Under the LRT rule, job $J_{q\xi}$, for which the total duration of all remaining operations is the maximum,

$$Q\left\{J_{q\xi}\right\} = \sum_{k=j}^{m} t_{q\xi k}, \tag{5}$$

is fed into machine M first.

2.3. The Rule TSW: SPT, LRT, and "Tense Job" Expectation Rules [13]

Of the many decisions that can be taken at any moment t, the one is taken that corresponds to the minimal change of the lower estimate of the criterion. This estimate is based on the sum of the time for performing the remaining operations. The approximation of the method consists in that the criterion estimate forecasted after each decision is regarded as the true value of the criterion, as a result of which, movement along only one branch of the decision tree takes place and the other branches are not memorized. Let us deal in more detail with realization of the preference rule.

At any moment t of freeing machine M, q "tense" job appears whose total time for performing the remaining operations (the time left for finishing an operation begun is not included in the total) is the maximum. The schedule for doing the remaining operations of the tense job depends strongly on the value of the criterion. If the tense job is in queue for machine S, it is fed in at moment t. If it has not yet been placed in the queue, compute the remaining time t_S^* for doing the tense job's operations till the one to be done on machine M, and compare it with time $t_S^{**} = \min_{\xi} t_{q\xi\, j\xi}$, the briefest operation of the jobs in queue for machine M (the SPT rule). If $t_S^{**} > t_S^*$, the machine stands idle till moment $t + t_S^*$. If $t_S^{**} \le t_S^*$, machine M is fed by job $J_{q\eta}$ ($t_S^{**} = t_{q\eta\, j\eta}$). Similarly, rule SPT can be substituted by rule LRT.

2.4. The Rule of Pairwise Comparison PC [10, 13]

For any two jobs J_i and J_k requiring to be operated on freed machine M, denote symbols O_{ij} and $O_{k\ell}$ for the routine operations, and symbols $O_{i,\,j+1},\ldots,O_{im}, O_{k,\,\ell+1},\ldots,O_{km}$ for the remaining operations of jobs J_i and J_k, respectively.

Let $t_{ij}, t_{k\ell}$ stand for the time to carry out operations $O_{ij}, O_{k\ell}$ and $\tau_{ij}, \tau_{k\ell}$ stand for the time to carry out all the remaining operations of jobs J_i and J_k, respectively. There exist two possible ways of feeding in the jobs: 1) First J_i, then J_k; 2) First J_k, then J_i. For the first way, we must calculate value $F(I) = t_{ij} + \max\left[\tau_{ij}, t_{k\ell} + \tau_{k\ell}\right]$, and for the second, $F(II) = t_{k\ell} + \max\left[\tau_{k\ell}, t_{ij} + \tau_{ij}\right]$. Say the first is preferable if there holds condition $F(I) < F(II)$. By comparing pairwise all jobs requiring operation on machine M, we can discover the job with the greatest priority.

3. Combined Preference Rules

3.1. The Rule of SPT and Feeding in Tense Jobs (Rule TS) [13–14]

A tense job is assigned to a freed machine if it must be done by moment t. Otherwise the next job is selected by the SPT rule. In keeping with the characteristic of rule (3), if at moment of time t jobs $J_{i_1},\ldots,J_{i_r}$ have completed operations $O_{i_1 j_1},\ldots,O_{i_r j_r}$ and have not yet begun their next operation, while jobs $J_{i_{i+1}},\ldots,J_{i_n}$ are still performing operations $O_{i_{r+1} j_{r+1}},\ldots,O_{i_n j_n}$, job $J_{i\xi}$ is considered a tense one if $\tau_{i\xi}(t) = \max\limits_{1\le k\le n} \tau_{i_k}(t)$, $\tau_{i_k}(t) = \sum\limits_{\gamma=j+1}^{m} t_{i_k \gamma}$ holds.

3.2. The Rule of PC and Starting a Tense Job (Rule TPC) [13–14]

A tense job is assigned first to a freed machine if it must be carried out. Otherwise, the job is chosen by rule PC.

3.3. The FIFO Rule [16]

The FIFO rule, in case a conflict situation appears, directs onto a machine the job from the queue waiting longest to be done, and consequently requiring to be done on that machine ahead of others. In other words, this is the principle of first come, first served.

Rules 4.1–4.3, following, employ as a parameter of the preference function the total time that the operations prior to the conflict situation waited in queues to be handled. Experimental calculations have shown that using such a parameter for several practical problems makes the preference function more efficient [14].

4. Total Time Heuristic Preference Algorithms

4.1. A Combination of LRT and Total Waiting Time [13–14]

Use symbol $T_t(J_i)$ for the total time of waiting in queues for job J_i at moment t. In other words, $T_t(J_i) = t - \sum_{j=1}^{j_i(t)} t_{ij}$, where $j_i(t)$ is the last operation of job J_i completed by moment t. Suppose that at moment t jobs $J_{i_1},...,J_{i_r}$ are to be done on one and the same machine. In such a case, the machine is fed job J_{i_ξ}, $1 \le \xi \le r$, for which the sum $T_t\ J_{i_\xi}\ + LRT\ J_{i_\xi}$ is maximized.

4.2. A Combination of FIFO, Total Waiting Time, and LRT [13, 14]

If at moment t jobs $J_{i_1},...,J_{i_r}$ are to be done on freed machine M, job J_{j_ξ}, first in the queue, is fed in. If there are several jobs of that kind $J_{j_1},...,J_{j_\ell}$, job J_{j_ξ}, $\xi = \overline{1,\ell}$, is fed in, the sum $T_t\ J_{j_\xi}\ + LRT\ J_{j_\xi}$ being maximized for it.

4.3. A Combination of Waiting for a Tense Job, Total Waiting Time, and LRT [14]

If at moment t jobs $J_{i_1},...,J_{i_r}$ are waiting to be done on the machine simultaneously, the tense job is found and moment t^* is determined for it to be fed in. If $\Delta t = t^* - t$ is no less than the time of the forthcoming operation for a part in the queue of jobs $J_{j_1},...,J_{j_\ell}$, job J_{j_ξ} is chosen and fed in, according to rule 4.1. Otherwise, the machine is idle and there is a wait for the tense job.

4.4. A Combination of Waiting for a Tense Job and FIFO

The procedure of preference rule 10 remains unchanged, except that of the set of jobs $J_{j_1},..,J_{j_\ell}$, the job first in queue is fed into the machine. If there are several such jobs, apply rule 4.1.

All the preference rules described above can be applied both in deterministic and randomized versions.

5. Using *mn*-Permutations

Note that, due to [15], the *mn*-permutations described can be regarded as priority vectors, the operation being given the greater priority the farther left the corresponding number in the *mn*-permutation is situated. This makes it possible to simulate various priority rules.

We will demonstrate how the most widespread priority rules, SPT and LRT, can be obtained by such an approach.

Transform initial matrix T as follows: Say

$$\tau_{ij} = \sum_{k=j+1}^{m} t_{ik},\ \tau_{im} = t_{im},\ i = \overline{1,n},\ j = \overline{1,m-1}.$$

Write all the elements of matrix $\|\tau_{ij}\|$ one after the other, in a row, so that element τ_{ij} stands at the $[(j-1)m+j]$-th place, after which write the row in descending order. Based on the indexes obtained of the elements of that row, the *mn*-permutations are determined with the help of the auxiliary algorithm of an actual schedule, described below. The schedule obtained will coincide with the one obtained by rule LRT (with accuracy of an ambiguous selection of operations by the LRT rule itself, when several operations have priority under that rule).

The SPT rule can be simulated as follows: Form two *mn*-length vectors $A1$ and $A2$, filling their elements consecutively from left to right. To determine the consecutive element $A1(k)$, take the leftmost element not crossed out in every row of matrix T and choose the minimum among them, which will then be crossed out of the matrix. Let the element crossed out have index (i, j). Then assume $A1(k)$ equals t_{ij}, and $A2(k) = (i-1)m + j$. Then, on the basis of the auxiliary algorithm, devise an actual schedule, using vector $A2$ as the initial mn-permutation. The resulting schedule coincides with the one obtained by rule SPT (as in the case of LRT, to an ambiguity accuracy resulting from the SIO rule itself).

The work of the auxiliary algorithm is as follows. Assume that in keeping with the procedure described in [15], at the k-th step of determining a possible schedule based on an *mn*-permutation, operation O_{ij} is assigned, with completion time equal to t_{ij}, and to be done on the M-th machine. Let job J_i be ready at moment τ_i to do operation O_{ij}. Look from left to right, beginning with moment τ_i, at all the idle intervals of the M-th machine. As soon as we find an idle interval with a duration more than zero, assign the corresponding operation O_{ij}. The starting moments of several operations can move to the right by a value no greater than t_{ij}.

This is how the last can be performed: Let operation O_{ij} be fed in during a certain idle interval of the i-th machine, and after the idleness, operation $O_{i_1 j_1}$ is to be done, to which the element at the j-th place, $j < k$, in the *mn*-permutation corresponds. In this case, the k-th element of the *mn*-permutation takes the j-th place, and the elements numbered $j, j+1, \ldots, k-1$ move one place to the right. Then return to the $(j+1)$-th element of the mn-permutation, and with it continue assigning the next operations.

From what has been said, it follows that an individual preference rule Q, the most efficient of the existing "rules bank" $\{Q_i\}$ must correspond as well to each individual technological matrix T. We believe such a pseudo-optimal rule should be built on the principles of self-edification with the help of a balanced probability combination of rules $\{Q_i\}$. We must make use of methods of search for the optimal balanced combination $\{\delta_i\}$ to make possible full use of the information provided by initial technological matrix T.

Suppose that there exists a data bank of preference rules consisting of d rules Q_p, $p = \overline{1,d}$, and we know how long it takes to do all the jobs with the help of each rule.

In keeping with the results of [11–15] say that we apply rule $Q = \sum_{p=1}^{d} \delta_p Q_p$ to technological matrix T if at the moment when a routine conflict situation arises, rule Q_p is applied to the queue of jobs with probability δ_p. Algorithmically, this means that every time, a rule Q_ξ, $\xi = \overline{1,d}$, is chosen, such that for its ordinal number relation

$$\sum_{p=1}^{\xi-1} \delta_p < \eta \le \sum_{p=1}^{\xi} \delta_p, \qquad (6)$$

holds, where η is the routine value of a random variable equi-distributed in interval $[0;1]$. Rule Q_ξ is applied (randomly or determined) to the queue of jobs according to the method described above.

Now we take up the development of an algorithm for forming an optimal set of weights $\{\delta_p\}$. The algorithm described below employs the procedure aimed at a random search, and consists of the following stages:

Stage I. Simulate d -dimensional vector $\vec{\Sigma}$ equi-distributed on a d -dimensional hyper-surface $\sum_{p=1}^{d} x_p = 1$, $0 \le x_p \le 1$, and denote the vector coordinates by symbols $\delta_1, \delta_2, ..., \delta_d$, [13].

Note that the order of enumerating coordinates δ_p in sequence $\{\delta_p\}$ does not change from now.

Stage II. Use the Monte Carlo method to simulate a random variable η, equi-distributed in interval $[0;1]$.

Stage III. Determine ordinal number $\xi \in \overline{1,d}$, such that relation (6) holds.

Stage IV. Draw up a calendar plan schedule for launching jobs on the machines in keeping with a periodical repetition of the Stages II $\Rightarrow$ III procedure and a set of values determined at Stage I for coordinates of vector $\vec{\Sigma}$ for each conflict situation. In other words, simulate the work of the system by deterministic or randomised rule $Q = \sum_{p=1}^{d} \delta_p Q_p$.

Stage V. Calculate the value of optimisation criterion K for a single realization of the model for drawing up a calendar schedule.

Stage VI. Repeat the procedure of Stages II-V a sufficiently large number of times (N times) to obtain representative statistics, and subsequent calculation of mathematical expectation $E[K]$ for simulated values $\delta_1, ..., \delta_d$. Denote the value obtained by symbol $E\{K[\delta_p Q_p]_T\}$, which signifies the estimate of the efficiency of the balanced preference rule $Q = \sum_{p=1}^{d} \delta_p Q_p$ for the initial technological matrix T.

Stage VII. Compare values $A = E\{K[\delta_p Q_p]\}$ and $B = \min_p \{K[Q_p]_T\}$. If inequality $A < B$ holds, it means we have determined (for initial matrix T) a better preference rule $Q_{d+1} = \sum_p \delta_p Q_p$ than those in the data bank, and we turn to Stage X of the algorithm. If $A \ge B$, go to the next stage.

Stage VIII. Counter η_1 of the number of consecutive unsuccessful tests at the initial point of search now functions. If counter $<\eta_1>$ shows no more than the limit number of unsuccessful tests $C^I_{\max}$, turn to Stage I. This means that the procedure of Stages I-VI is again repeated, in order to provide a better balance rule Q_{d+1} . If $<\eta_1> > C^I_{\max}$, transfer to the next stage of the algorithm.

Stage IX. For the initial point of a local search, take vector-point $\vec{\Sigma}^{(1)} = \delta_p^{(1)}$, where $\delta_p^{(1)} = 0$ when $p \ne \gamma$, and $\delta_p^{(1)} = 1$ when $p = \gamma$, $\gamma \in \overline{1,d}$, $E\{K[Q_\gamma]_T\} = B$. Further, turn to Stage XI, preliminarily sending the coordinate values of vector-point $\vec{\Sigma}^{(1)}$ and of $B = K_{\min}$ to a special array Z.

Stage X. Take vector-point $\vec{\Sigma}^{(1)} = \{\delta_p^{(1)}\}$ as the initial point of the local random search for an extremum, with $E\left[K\left[\delta_p^{(1)} Q_p\right]_T\right] = A$. The coordinates of vector-point $\vec{\Sigma}^{(1)}$ and value $A = K_{\min}$ are sent to array Z.

Stage XI. Make a local random search by

$$\vec{\Sigma}^{(2)} = \vec{\Sigma}^{(1)} + \vec{X}\ , \tag{7}$$

where $\vec{X}$ is a random d-dimensional vector-point, whose coordinates x_p satisfy relations $\sum_{p=1}^{d} x_p = 0$ and $\sum_{p=1}^{d} x_p^2 = D$, where D is a previously accepted length of the search step.

Stage XII. Correct vector-point $\vec{\Sigma}^{(2)}$ if it leads beyond the region of $0 \le \delta_p^{(2)} \le 1$, $p = \overline{1,d}$. There are several ways to effect that correction, in particular by reducing search step D (by halving, for instance) till restrictions $0 \le \delta_p^{(2)} \le 1$ are satisfied.

Take note of the fact that the search procedure can also be followed in other ways, including heuristic elements, for example, by relation

$$\delta_p^{(2)*} = \delta_p^{(1)} + D \cdot sign\,\alpha_p\ , \tag{8}$$

where α_p, $p = \overline{1,d}$, are independent values of a random variable equi-distributed in interval $[-1;+1]$, and subsequent correction

$$\delta_p^{(2)**} = \begin{cases} \delta_p^{(2)*} & when\ 0 \le \delta_p^{(2)*} \le 1 \\ 1 & when\ 1 < \delta_p^{(2)*} \\ 0 & when\ 0 > \delta_p^{(2)*} \end{cases}\ , \tag{9}$$

$$\delta_p^{(2)*} = \delta_p^{(2)**} \cdot \frac{1}{\sum_p \delta_p^{(2)**}}\ . \tag{10}$$

Stage XIII. Perform the procedure of Stages II-VI (N times), preliminarily clearing counter η_1 and subsequently calculating value $E\ K\ \delta_p^{(2)} Q_p\ _T$.

Stage XIV. Compare value $E\ K\ \delta_p^{(2)} Q_p\ _T$ with the contents of the cell in array Z, where the minimal value of criterion $K_{\min}$ is stored. If the new estimate is less than value $K_{\min}$, transfer to Stage XVI. If not, move on to the next stage.

Stage XV. Counter η_2 of the consecutive unsuccessful search steps now functions. If $<\eta_2>$ is less than limit number $C_{\max}^{II}$, transfer to Stage XI for a repeated realization of a random search step from the previous vector-point $\vec{\Sigma}^{(1)}$. Otherwise, turn to Stage XVIII.

Stage XVI. Clear counter η_2 and then compare values

$$\Delta_1 = \left| \frac{K\ \delta_p^{(2)} Q_p\ _T - K_{\min}}{K_{\min}} \right| \text{ and } \Delta_{\min},$$

where $\Delta_{\min}$ is a pregiven relative error of the convergence of the optimisation criterion for a local random search. If $\Delta_1 \le \Delta_{\min}$, turn to Stage XIX; otherwise advance to Stage XVII.

Stage XVII. Send values of vector-point $\vec{\Sigma}^{(2)}$ and $E\ K\ \delta_p^{(2)} Q_p\ _T$ to array Z, subsequently transferring to Stage XI. In other words, the new vector-point $\vec{\Sigma}^{(2)}$ takes the place of the previous one $\vec{\Sigma}^{(2)} \Rightarrow \vec{\Sigma}^{(1)}$, and the random search is carried out from the new point. Value $E\ K\ \delta_p^{(2)} Q_p\ _T$ now becomes minimal value $K_{\min}$.

Stage XVIII. Choose and print the local optimum, an optimal set of balance coefficients $\delta_p^{(opt)}$ from array Z, and the corresponding value $K_{\min}$. After this, proceed to Stage XX.

Stage XIX. Choose and print the local optimum $\delta_p^{(2)}$ and the value of optimisation criterion $K_{\min} = E\ K\ \delta_p^{(2)} Q_p\ _T$. Further, turn to the next stage of the algorithm.

Stage XX. Counter η_3 of the determined local optimums works. If $<\eta_3>$ exceeds the limit number $C_{\max}^{III}$, proceed to Stage XXII. Otherwise, go to the next stage of the algorithm.

Stage XXI. Clear counters η_1 and η_2 and array Z. Further, return to Stage I.

Stage XXII. Choose an overall optimum, set $\{\delta_p^*\}$, furnishing a minimum of local optimums built on Stages XVIII–XIX, after which the work of the algorithm terminates.

Note that such an adaptive algorithm, as we see it, especially efficient in case it is necessary to perform operative control of the process of feeding jobs to machines at random durations of separate operations O_{ij}, i.e., at the stage of actual production. In such a case, at moment t for each conflict situation, jobs must be chosen for machines according to the optimal set $\{\delta_p^*\}$ determined at moment t. The durations t_{ij} of the remaining operations O_{ij} are equated to their mathematical expectations, and the reduced matrix T_t, including only the operations still to be done, is taken as the technological matrix T. If another conflict situation arises, we do the same, each time acting on decisions taken, i.e., working out the appropriate control action.

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NEW APPROACH TO PATTERN RECOGNITION VIA COMPARISON OF MAXIMUM SEPARATIONS

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Fisher's linear discriminant analysis is a widely used multivariate statistical technique with two closely related goals: discrimination and classification. The technique is very popular among users of discriminant analysis. Some of the reasons for this are its simplicity and un-necessity of strict assumptions. In its original form, proposed by Fisher, the method assumes equality of population covariance matrices, but does not explicitly require multivariate normality. However, optimal classification performance of Fisher's discriminant function can only be expected when multivariate normality is present as well, since only good discrimination can ensure good allocation. In practice, we often are in need of analysing input data samples, which are not adequate for Fisher's classification rule, such that the distributions of the groups are not multivariate normal or covariance matrices of those are different or there are strong multi-nonlinearities.

This paper proposes a new approach to pattern recognition based on comparison of maximum separations in the input data samples, where maximum separations are determined via Fisher's separation criterion. The approach represents the improved pattern recognition procedure that allows one to take into account the cases which are not adequate for Fisher's classification rule. Moreover, it allows one to classify sets of multivariate observations, where each of the sets contains more than one observation. For the cases, which are adequate for Fisher's classification rule, the proposed approach gives the results similar to that of Fisher's classification rule. Illustrative examples are given.

Keywords: *Fisher's separation criterion, input data samples, maximum separation, pattern recognition*

1. Introduction

Humans have developed highly sophisticated skills for sensing their environment and taking actions according to what they observe, e.g., recognizing a face, understanding spoken words, reading handwriting, distinguishing fresh food from its smell. We would like to give similar capabilities to machines. A pattern is an entity, vaguely defined, that could be given a name, e.g., fingerprint image, handwritten word, human face, speech signal, etc. Pattern recognition is the study of how machines can observe the environment, learn to distinguish patterns of interest, make sound and reasonable decisions about the categories of the patterns. We are often influenced by the knowledge of how patterns are modelled and recognized in nature when we develop pattern recognition algorithms. Research on machine perception also helps us gain deeper understanding and appreciation for pattern recognition systems in nature. Yet, we also apply many techniques that are purely numerical and do not have any correspondence in natural systems. Pattern recognition techniques find applications in many areas: machine learning, statistics, mathematics, computer science, biology, etc. There are many sub-problems in the design process. Many of these problems can indeed be solved. More complex learning, searching and optimisation algorithms are developed with advances in computer technology. There remain many fascinating unsolved problems.

The aim of pattern recognition is to classify data (patterns) based on either a priori knowledge or on statistical information extracted from the patterns. The patterns to be classified are usually groups of measurements or observations, defining points in an appropriate multidimensional space. Many pattern recognition methods can be decomposed into two stages: discrimination followed by classification.

In some cases, the decomposition is explicit while in others it is a matter of interpretation. Discrimination and classification represent multivariate techniques concerned with separating distinct sets of objects (or observations) and allocating new objects (observations) to previously defined groups. There exist situations in which one may be interested in (1) discrimination: separating, say, two classes of objects or (2) classification: assigning a new object to one of two classes (or both).

The most popular separation criterion of establishing rules for discrimination and classification of patterns is the Fisher's discriminant (separation) ratio. Fisher's idea was to transform the ($p \geq 2$) multivariate observations $\mathbf{x}$ to univariate observations y such that the y's derived from populations π_1 and π_2 were separated as much as possible. Fisher suggested taking linear combinations of $\mathbf{x}$ to create y's because they are simple enough functions of the $\mathbf{x}$ to be handled easily. Fisher's approach does not assume that the populations are normal. It does, however, implicitly assume that the population covariance matrices are equal, because a pooled estimate of the common covariance matrix is used. A fixed linear combination of the $\mathbf{x}$'s takes the values $y_{11}, y_{12}, \ldots, y_{1n_1}$ for the n_1 observations of the $\mathbf{x}^{n1} = (\mathbf{x}_{11}, \ldots, \mathbf{x}_{1n_1})$ from the first population π_1 and the values $y_{21}, y_{22}, \ldots, y_{2n_2}$ for the n_2 observations of the sample $\mathbf{x}^{n2} = (\mathbf{x}_{21}, \ldots, \mathbf{x}_{2n_2})$ from the second population π_2. The separation of these two sets of univariate y's is assessed in terms of the difference between the sample means $\bar{y}_1$ and $\bar{y}_2$ expressed in standard deviation units. The separation criterion proposed by Fisher is given by

$$I(\mathbf{u}) = \frac{(\bar{y}_1 - \bar{y}_2)^2}{s_y^2} = \frac{(\mathbf{u}'\bar{\mathbf{x}}_1 - \mathbf{u}'\bar{\mathbf{x}}_2)^2}{\mathbf{u}'\mathbf{S}_{\text{pooled}}\mathbf{u}} = \frac{[\mathbf{u}'(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)]^2}{\mathbf{u}'\mathbf{S}_{\text{pooled}}\mathbf{u}}, \tag{1}$$

and has to be maximized with respect to $\mathbf{u}$ (transformation vector), where

$$s_y^2 = \frac{\sum_{j=1}^{n1}(y_{1i} - \bar{y}_1)^2 + \sum_{j=1}^{n1}(y_{2i} - \bar{y}_2)^2}{n_1 + n_2 - 2} = \frac{\sum_{j=1}^{n1}(\mathbf{u}'\mathbf{x}_{1j} - \mathbf{u}'\bar{\mathbf{x}}_1)^2 + \sum_{j=1}^{n2}(\mathbf{u}'\mathbf{x}_{2j} - \mathbf{u}'\bar{\mathbf{x}}_2)^2}{n_1 + n_2 - 2}$$

$$= \frac{\sum_{j=1}^{n1}[\mathbf{u}'(\mathbf{x}_{1j} - \bar{\mathbf{x}}_1)]'[\mathbf{u}'(\mathbf{x}_{1j} - \bar{\mathbf{x}}_1)] + \sum_{j=1}^{n2}[\mathbf{u}'(\mathbf{x}_{2j} - \bar{\mathbf{x}}_2)]'[\mathbf{u}'(\mathbf{x}_{2j} - \bar{\mathbf{x}}_2)]}{n_1 + n_2 - 2}$$

$$= \frac{\sum_{j=1}^{n1}[(\mathbf{x}_{1j} - \bar{\mathbf{x}}_1)'\mathbf{u}]'[(\mathbf{x}_{1j} - \bar{\mathbf{x}}_1)'\mathbf{u}] + \sum_{j=1}^{n2}[(\mathbf{x}_{2j} - \bar{\mathbf{x}}_2)'\mathbf{u}]'[(\mathbf{x}_{2j} - \bar{\mathbf{x}}_2)'\mathbf{u}]}{n_1 + n_2 - 2}$$

$$= \frac{\sum_{j=1}^{n1}\mathbf{u}'[(\mathbf{x}_{1j} - \bar{\mathbf{x}}_1)(\mathbf{x}_{1j} - \bar{\mathbf{x}}_1)']\mathbf{u} + \sum_{j=1}^{n2}\mathbf{u}'[(\mathbf{x}_{2j} - \bar{\mathbf{x}}_2)(\mathbf{x}_{2j} - \bar{\mathbf{x}}_2)']\mathbf{u}}{n_1 + n_2 - 2}$$

$$= \frac{\mathbf{u}'[(n_1 - 1)\mathbf{S}_1]\mathbf{u} + \mathbf{u}'[(n_2 - 1)\mathbf{S}_2]\mathbf{u}}{n_1 + n_2 - 2} = \mathbf{u}'\mathbf{S}_{\text{pooled}}\mathbf{u} \tag{2}$$

is the pooled estimate of the variance. The sample means vectors and covariance matrices are determined by

$$\bar{\mathbf{x}}_1 = \frac{1}{n_1}\sum_{j=1}^{n_1}\mathbf{x}_{1j}, \qquad \mathbf{S}_1 = \frac{1}{n_1 - 1}\sum_{j=1}^{n_1}(\mathbf{x}_{1j} - \bar{\mathbf{x}}_1)(\mathbf{x}_{1j} - \bar{\mathbf{x}}_1)' \tag{3}$$

and

$$\bar{\mathbf{x}}_2 = \frac{1}{n_2}\sum_{j=1}^{n_2}\mathbf{x}_{2j}, \qquad \mathbf{S}_2 = \frac{1}{n_2 - 1}\sum_{j=1}^{n_2}(\mathbf{x}_{2j} - \bar{\mathbf{x}}_2)(\mathbf{x}_{2j} - \bar{\mathbf{x}}_2)'. \tag{4}$$

Since it is assumed implicitly that the populations π_1 and π_2 have the same covariance matrix $\mathbf{\Sigma}$, the sample covariance matrices $\mathbf{S}_1$ and $\mathbf{S}_2$ are combined (pooled) to derive a single unbiased estimate of $\mathbf{\Sigma}$,

$$\mathbf{S}_{\text{pooled}} = \frac{(n_1 - 1)\mathbf{S}_1 + (n_2 - 1)\mathbf{S}_2}{n_1 + n_2 - 2}. \tag{5}$$

Theorem 1. (*Separation criterion: maximization*). Let $\mathbf{S}_{\text{pooled}}$ be positive definite and $(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)$ be a given vector. Then the optimal transformation vector is given by

$$\mathbf{u}^* = \arg \max_{\mathbf{u} \neq \mathbf{0}} I(\mathbf{u}) = \arg \max_{\mathbf{u} \neq \mathbf{0}} \frac{[\mathbf{u}'(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)]^2}{\mathbf{u}'\mathbf{S}_{\text{pooled}}\mathbf{u}} = \mathbf{S}_{\text{pooled}}^{-1}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) \tag{6}$$

with the maximum attained

$$I(\mathbf{u}^*) = \max_{\mathbf{u} \neq \mathbf{0}} \frac{[\mathbf{u}'(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)]^2}{\mathbf{u}'\mathbf{S}_{\text{pooled}}\mathbf{u}} = (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{\text{pooled}}^{-1}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2). \tag{7}$$

Proof. By the extended Cauchy-Schwarz inequality [1],

$$[\mathbf{u}'(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)]^2 \leq [\mathbf{u}'\mathbf{S}_{\text{pooled}}\mathbf{u}][(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{\text{pooled}}^{-1}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)]. \tag{8}$$

Because $\mathbf{u} \neq 0$ and $\mathbf{S}_{\text{pooled}}$ is positive definite, $\mathbf{u}'\mathbf{S}_{\text{pooled}}\mathbf{u} > 0$. Dividing both sides of the inequality (8) by the positive scalar $\mathbf{u}'\mathbf{S}_{\text{pooled}}\mathbf{u}$ yields the upper bound

$$\frac{[\mathbf{u}'(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)]^2}{\mathbf{u}'\mathbf{S}_{\text{pooled}}\mathbf{u}} \leq (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{\text{pooled}}^{-1}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2). \tag{9}$$

Taking the maximum over $\mathbf{u}$ gives Equation (7) because the bound is attained for $\mathbf{u} = \mathbf{S}_{\text{pooled}}^{-1}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)$. This completes the proof.

Fisher's linear discriminant analysis has been successfully used as dimensionality reduction technique to many classification problems, such as face recognition and multimedia information retrieval. The Fisher discriminant criterion is the benchmark for the linear discrimination in multidimensional space [2]. The criterion purpose of the Fisher linear discriminant for pattern analysis is to find an optimal discriminant direction based on the Fisher criterion so that the projected set of training samples on it has the maximal ratio of between-class distance to within-class distance [3–4]. Sammon extended the Fisher linear discriminant method to the optimal discriminant plane in 1970 [5]. Then Foley and Sammon [6] further extended this in 1975 and proposed the optimal set of discriminant vectors by which the well-known Foley-Sammon Transform (FST) can be constituted. Their important result has attracted many researchers' attention in the field of pattern recognition [7–8] and has been used in many pattern classification applications [9–10].

2. Fisher's Approach to Pattern Recognition

2.1. Case of Two Populations

In this case, Fisher's approach to pattern recognition is as follows:

Discrimination. At this step, the optimal transformation vector (Fisher's solution), which allows one to maximally separate the two populations (π_1 and π_2), is determined by (6). The maximum separation in the samples is given by (7).

Classification. At this step, Fisher's solution to the separation problem (Fisher's discriminant function) is used to classify new observations as follows:

Allocate $\mathbf{x}_0$ to π_1 if

$$y_0 = (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{\text{pooled}}^{-1}\mathbf{x}_0 \geq \frac{\bar{y}_1 + \bar{y}_2}{2} = \frac{(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{\text{pooled}}^{-1}(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2)}{2}; \tag{10}$$

Allocate $\mathbf{x}_0$ to π_2 if

$$y_0 = (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{\text{pooled}}^{-1}\mathbf{x}_0 < \frac{\bar{y}_1 + \bar{y}_2}{2} = \frac{(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{\text{pooled}}^{-1}(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2)}{2}. \tag{11}$$

Thus, the allocation rule based on Fisher's discriminant function to classify $\mathbf{x}_0$ is given by

$$(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{\text{pooled}}^{-1}\mathbf{x}_0 \begin{cases} \geq [(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{\text{pooled}}^{-1}(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2)]/2, & \text{then} \quad \mathbf{x}_0 \in \pi_1, \\ < [(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{S}_{\text{pooled}}^{-1}(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2)]/2, & \text{then} \quad \mathbf{x}_0 \in \pi_2. \end{cases} \tag{12}$$

A pictorial representation of Fisher's procedure for two *p*-variate populations with *p*=2 is given in Figure 1.

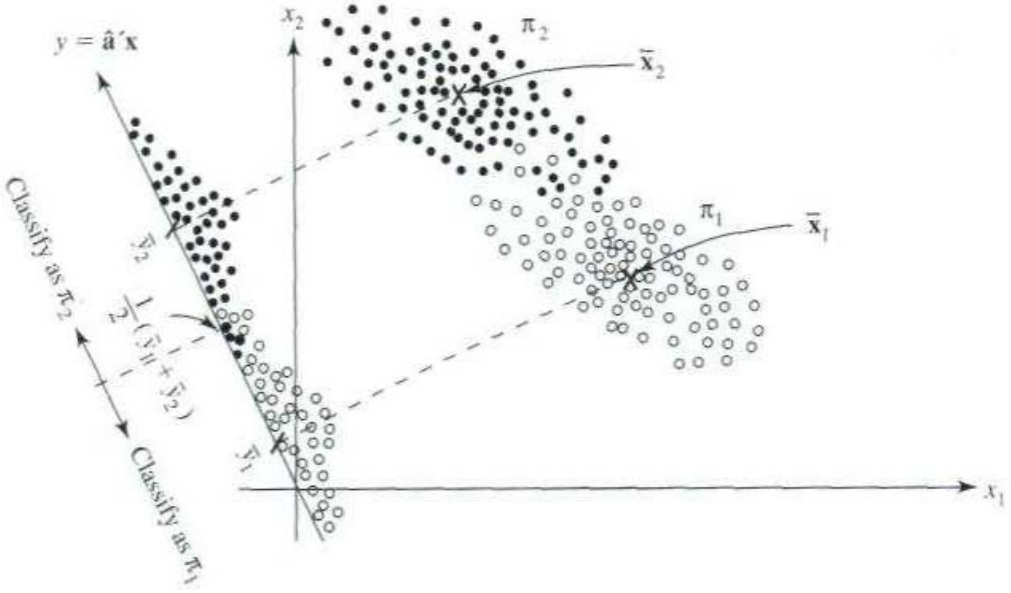

Figure 1. A pictorial representation of Fisher's procedure for two p-variate populations with p = 2

The misclassification probabilities are shown on Figure 2.

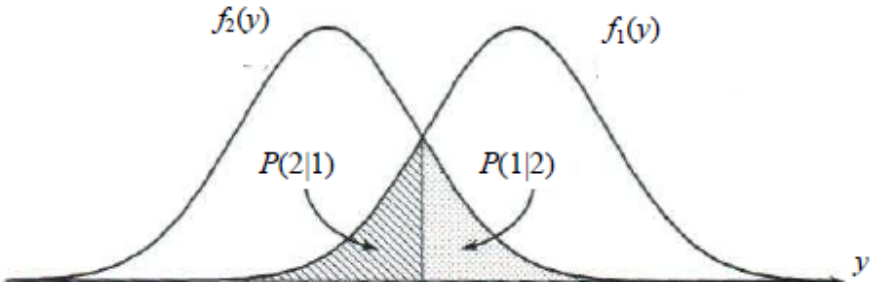

Figure 2. The misclassification probabilities based on Y

Thus, Fisher's approach to pattern recognition has optimality properties (in particular, the minimum total probability of misclassification) only if the underlying distributions of the groups of univariate observations (in terms of the random variable *Y*) obtained via transformation of the groups of multivariate observations (in terms of the random variable **x**) are symmetrical with equal variances.

2.2. Case of Several Populations

In this case, Fisher's approach to pattern recognition consists in the following:

Discrimination. At this step, the optimal transformation vector (Fisher's solution), which allows one to maximally separate the ($m > 2$) populations ($\pi_1, \ldots, \pi_m$), is determined by

$$\mathbf{u}^* = \arg\max_{\mathbf{u} \neq \mathbf{0}} I(\mathbf{u}), \tag{13}$$

where

$$I(\mathbf{u}) = \frac{\mathbf{u}'\mathbf{B}\mathbf{u}}{\mathbf{u}'\mathbf{W}\mathbf{u}} \tag{14}$$

represents the criterion of separation in the input data samples,

$$\mathbf{B} = \sum_{i=1}^{m} n_i (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \tag{15}$$

is the *sample between groups matrix* which includes the sample sizes n_i, sample mean vectors

$$\bar{\mathbf{x}}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} \mathbf{x}_{ij}, \quad i = 1, \ldots, m, \tag{16}$$

and the "overall average"

$$\bar{\mathbf{x}} = \sum_{i=1}^{m} n_i \bar{x}_i \Big/ \sum_{i=1}^{m} n_i, \tag{17}$$

which is the $p \times 1$ vector average taken over all the sample observations in the set of input data samples,

$$\mathbf{W} = \sum_{i=1}^{m} \sum_{j=1}^{n_i} (\mathbf{x}_{ij} - \bar{\mathbf{x}}_i)(\mathbf{x}_{ij} - \bar{\mathbf{x}}_i)' = \sum_{i=1}^{m} n_i - m \ \mathbf{S}_{\text{pooled}} \tag{18}$$

is the *sample within group matrix*.

The maximum separation in the input data samples can be obtained as follows. Maximizing I (**u**) by taking the derivative with respect to **u** and setting it to 0,

$$\frac{dI(\mathbf{u})}{d\mathbf{u}} = \frac{d}{d\mathbf{u}} \ \frac{\mathbf{u}'\mathbf{B}\mathbf{u}}{\mathbf{u}'\mathbf{W}\mathbf{u}} \ = 0, \tag{19}$$

we have

$$(\mathbf{W}^{-1}\mathbf{B} - \lambda\mathbf{I})\mathbf{u} = 0, \tag{20}$$

where

$$\lambda = \frac{\mathbf{u}'\mathbf{B}\mathbf{u}}{\mathbf{u}'\mathbf{W}\mathbf{u}}. \tag{21}$$

Thus, we deal with a standard eigenvalue problem. Now, we must solve

$$\left|\mathbf{W}^{-1}\mathbf{B} - \lambda\mathbf{I}\right| = 0 \tag{22}$$

for the r nonzero eigenvalues ($\lambda_1, \ldots, \lambda_r$) of $\mathbf{W}^{-1}\mathbf{B}$, where $r \le \min(m - 1, p)$. Then the corresponding normalized eigenvectors are obtained by solving

$$(\mathbf{W}^{-1}\mathbf{B} - \lambda_k\mathbf{I})\mathbf{u}_k^* = 0, \quad k = 1, \ldots, r, \tag{23}$$

and scaling the results such that

$$(\mathbf{u}_k^*)'\mathbf{S}_{\text{pooled}}\mathbf{u}_k^* = 1, \quad k = 1, \ldots, r, \tag{24}$$

Thus, the maximum separation in the input data samples based on the criterion (14) is given by

$$\frac{\sum_{k=1}^{r} (\mathbf{u}_k^*)'\mathbf{B}\mathbf{u}_k^*}{\sum_{k=1}^{r} (\mathbf{u}_k^*)'\mathbf{W}\mathbf{u}_k^*} = \frac{\sum_{k=1}^{r} (\mathbf{u}_k^*)' \sum_{i=1}^{m} n_i (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \ \mathbf{u}_k^*}{\sum_{k=1}^{r} (\mathbf{u}_k^*)' \sum_{i=1}^{m} \sum_{j=1}^{n_i} (\bar{\mathbf{x}}_{ij} - \bar{\mathbf{x}}_i)(\bar{\mathbf{x}}_{ij} - \bar{\mathbf{x}}_i)' \ \mathbf{u}_k^*}. \tag{25}$$

Classification. At this step, Fisher's solution to the separation problem (Fisher's discriminant function) is used to classify new observations as follows:

Allocate $\mathbf{x}_0$ to π_l, $l \in \{1, \ldots, m\}$ if

$$\sum_{k=1}^{r}(y_{0k}-\bar{y}_{lk})^2=\sum_{k=1}^{r}[(\mathbf{u}_k^*)'(\mathbf{x}_0-\bar{\mathbf{x}}_l)]^2 \le \sum_{k=1}^{r}[(\mathbf{u}_k^*)'(\mathbf{x}_0-\bar{\mathbf{x}}_i)]^2 \quad \text{for all } i \in \{1, \ldots, m\},\ i \ne l, \tag{26}$$

where $\mathbf{u}_k^*$ is defined in (21), $\bar{y}_{lk}=(\mathbf{u}_k^*)'\bar{\mathbf{x}}_l$ and $r \le \min(m-1, p)$.

3. Pattern Recognition Based on Comparing of Maximum Separations in Data Samples

3.1. Case of Two Populations

Discrimination. At this step, based on (7) we determine the maximum separations in the two input data samples taking into account a new observation $\mathbf{x}_0$.

Classification. In order to classify a new observation $\mathbf{x}_0$ to one of the two populations (π_1, π_2), we take decision based on comparison of the maximum separations in the two source data samples as follows:

Allocate $\mathbf{x}_0$ to π_1,

if the maximum separation $(\bar{\mathbf{x}}_{1(0)}-\bar{\mathbf{x}}_2)'\mathbf{S}_{\text{pooled}(1)}^{-1}(\bar{\mathbf{x}}_{1(0)}-\bar{\mathbf{x}}_2)$ ($\mathbf{x}_0$ has been added to the sample $\mathbf{x}^{n_1}$ from π_1)

$\ge$ the maximum separation $(\bar{\mathbf{x}}_1-\bar{\mathbf{x}}_{2(0)})'\mathbf{S}_{\text{pooled}(2)}^{-1}(\bar{\mathbf{x}}_1-\bar{\mathbf{x}}_{2(0)})$ ($\mathbf{x}_0$ has been added to the sample $\mathbf{x}^{n_2}$ from π_2), (27)

where

$$\bar{\mathbf{x}}_{1(0)}=\frac{1}{n_1+1}\sum_{j=0}^{n_1}\mathbf{x}_{1i}, \qquad \mathbf{S}_{1(0)}=\frac{1}{n_1}\sum_{j=0}^{n_1}(\mathbf{x}_{1i}-\bar{\mathbf{x}}_{1(0)})(\mathbf{x}_{1i}-\bar{\mathbf{x}}_{1(0)})', \qquad \mathbf{x}_{10}\equiv\mathbf{x}_0, \tag{28}$$

$$\bar{\mathbf{x}}_{2(0)}=\frac{1}{n_2+1}\sum_{j=0}^{n_2}\mathbf{x}_{2i}, \qquad \mathbf{S}_{2(0)}=\frac{1}{n_2}\sum_{j=0}^{n_2}(\mathbf{x}_{2i}-\bar{\mathbf{x}}_{2(0)})(\mathbf{x}_{2i}-\bar{\mathbf{x}}_{2(0)})', \qquad \mathbf{x}_{20}\equiv\mathbf{x}_0, \tag{29}$$

and

$$\mathbf{S}_{\text{pooled}(1)}=\frac{n_1\mathbf{S}_{1(0)}+(n_2-1)\mathbf{S}_2}{n_1+n_2-1}, \qquad \mathbf{S}_{\text{pooled}(2)}=\frac{(n_1-1)\mathbf{S}_1+n_2\mathbf{S}_{2(0)}}{n_1+n_2-1}. \tag{30}$$

Allocate $\mathbf{x}_0$ to π_2,

if the maximum separation $(\bar{\mathbf{x}}_{1(0)}-\bar{\mathbf{x}}_2)'\mathbf{S}_{\text{pooled}(1)}^{-1}(\bar{\mathbf{x}}_{1(0)}-\bar{\mathbf{x}}_2)$ ($\mathbf{x}_0$ has been added to the sample $\mathbf{x}^{n_1}$ from π_1)

$<$ the maximum separation $(\bar{\mathbf{x}}_1-\bar{\mathbf{x}}_{2(0)})'\mathbf{S}_{\text{pooled}(2)}^{-1}(\bar{\mathbf{x}}_1-\bar{\mathbf{x}}_{2(0)})$ ($\mathbf{x}_0$ has been added to the sample $\mathbf{x}^{n_2}$ from π_2). (31)

Thus, the allocation rule based on comparison of the maximum separations in the two input data samples to classify $\mathbf{x}_0$ is given by

$$\frac{\max\limits_{\mathbf{u}\ne\mathbf{0}}\dfrac{[\mathbf{u}'(\bar{\mathbf{x}}_{1(0)}-\bar{\mathbf{x}}_2)]^2}{\mathbf{u}'\mathbf{S}_{\text{pooled}(1)}\mathbf{u}}}{\max\limits_{\mathbf{u}\ne\mathbf{0}}\dfrac{[\mathbf{u}'(\bar{\mathbf{x}}_1-\bar{\mathbf{x}}_{2(0)})]^2}{\mathbf{u}'\mathbf{S}_{\text{pooled}(2)}\mathbf{u}}}=\frac{(\bar{\mathbf{x}}_{1(0)}-\bar{\mathbf{x}}_2)'\mathbf{S}_{\text{pooled}(1)}^{-1}(\bar{\mathbf{x}}_{1(0)}-\bar{\mathbf{x}}_2)}{(\bar{\mathbf{x}}_1-\bar{\mathbf{x}}_{2(0)})'\mathbf{S}_{\text{pooled}(2)}^{-1}(\bar{\mathbf{x}}_1-\bar{\mathbf{x}}_{2(0)})}=\frac{\bar{y}_{1(0)}-\bar{y}_2}{\bar{y}_1-\bar{y}_{2(0)}}\begin{array}{ll}\ge 1, & \text{then } \mathbf{x}_0\in\pi_1,\\ <1, & \text{then } \mathbf{x}_0\in\pi_2.\end{array} \tag{32}$$

3.2. Case of $Several Populations

3.2.1. Version I (Paired-comparison procedure)

Discrimination. At this step, based on (7) we determine the maximum separations in the input data samples taking into account a new observation $\mathbf{x}_0$.

Classification. In order to classify $\mathbf{x}_0$ to one of several, say, m ($m > 2$) populations, we use the ratio of the maximum separations in the input data samples as follows. At first, any two data samples from populations, say, π_k and π_r, respectively, are considered.

The allocation rule based on the ratio of the maximum separations in the above data samples to classify $\mathbf{x}_0$ is given by

$$\begin{aligned} \frac{[(\bar{\mathbf{x}}_{k(0)} - \bar{\mathbf{x}}_r)'\mathbf{S}^{-1}_{\text{pooled}(k)}(\bar{\mathbf{x}}_{k(0)} - \bar{\mathbf{x}}_r) \mid (\mathbf{x}_0, \mathbf{x}^{n_k}), \mathbf{x}^{n_r}]}{[(\bar{\mathbf{x}}_k - \bar{\mathbf{x}}_{r(0)})'\mathbf{S}^{-1}_{\text{pooled}(r)}(\bar{\mathbf{x}}_k - \bar{\mathbf{x}}_{r(0)}) \mid \mathbf{x}^{n_k}, (\mathbf{x}_0, \mathbf{x}^{n_r})]} &\geq 1, \quad \text{then} \quad \mathbf{x}_0 \in \pi_k, \\ &< 1, \quad \text{then} \quad \mathbf{x}_0 \in \pi_r. \end{aligned} \tag{33}$$

where

$$\bar{\mathbf{x}}_{k(0)} = \frac{1}{n_k + 1}\sum_{i=0}^{n_k} \mathbf{x}_{ki}, \qquad \mathbf{S}_{k(0)} = \frac{1}{n_k}\sum_{i=0}^{n_k} (\mathbf{x}_{ki} - \bar{\mathbf{x}}_{k(0)})(\mathbf{x}_{ki} - \bar{\mathbf{x}}_{k(0)})', \qquad \mathbf{x}_{k0} \equiv \mathbf{x}_0, \tag{34}$$

$$\bar{\mathbf{x}}_{r(0)} = \frac{1}{n_r + 1}\sum_{i=0}^{n_r} \mathbf{x}_{ri}, \qquad \mathbf{S}_{r(0)} = \frac{1}{n_r}\sum_{i=0}^{n_r} (\mathbf{x}_{ri} - \bar{\mathbf{x}}_{r(0)})(\mathbf{x}_{ri} - \bar{\mathbf{x}}_{r(0)})', \qquad \mathbf{x}_{r0} \equiv \mathbf{x}_0, \tag{35}$$

and

$$\mathbf{S}_{\text{pooled}(k)} = \frac{n_k\mathbf{S}_{k(0)} + (n_r - 1)\mathbf{S}_{\text{r}}}{n_k + n_r - 1}, \qquad \mathbf{S}_{\text{pooled(r)}} = \frac{(n_k - 1)\mathbf{S}_k + n_r\mathbf{S}_{r(0)}}{n_k + n_r - 1}. \tag{36}$$

If $\mathbf{x}_0 \in \pi_k$, then the population π_r is eliminated from further consideration. If $(m - 1)$ populations are so eliminated, then the remaining population (say, kth) is the one to which the observation $\mathbf{x}_0$ being classified belongs.

3.2.2. Version II

Discrimination. At this step, based on (25) we determine the maximum separations in the input data samples taking into account a new observation $\mathbf{x}_0$.

Classification. At this step, the maximum separation in the input data samples is used to classify $\mathbf{x}_0$ as follows:

Allocate $\mathbf{x}_0$ to π_l, $l \in \{1, \ldots, m\}$ if

$$\frac{\sum_{k=1}^{r} (\mathbf{u}^*_{k(l)})'\mathbf{B}_l\mathbf{u}^*_{k(l)}}{\sum_{k=1}^{r} (\mathbf{u}^*_{k(l)})'\mathbf{W}_l\mathbf{u}^*_{k(l)}} \geq \frac{\sum_{k=1}^{r} (\mathbf{u}^*_{k(s)})'\mathbf{B}_i\mathbf{u}^*_{k(s)}}{\sum_{k=1}^{r} (\mathbf{u}^*_{k(s)})'\mathbf{W}_i\mathbf{u}^*_{k(s)}} \quad \text{for all } s \in \{1, \ldots, m\},\ s \neq l, \tag{37}$$

where $\mathbf{x}_0$ has been added to the sample $\mathbf{x}^{n_l}$ from π_l, $\mathbf{u}^*_{k(l)}$ ($k = 1, \ldots, r$) are defined via (12), and $r \leq \min(m - 1, p)$.

4. Examples

4.1. Example 1

This example is adapted from a study [11] concerned with the detection of hemophilia A carriers. To construct a procedure for detecting potential hemophilia A carriers, blood samples were assayed for two groups of women and measurements on the two variables,

$X_1 = \log_{10}$ (AHF activity) and $X_2 = \log_{10}$ (AHF-like antigen) (38)

recorded (see Table 1). ("AHF" denotes antihemophilic factor.)

Table 1. Hemophilia data

Group 1: Noncarriers (π_1)			Group 2: Obligatory carriers (π_2)		
No.	$\log_{10}$(AHF activity)	$\log_{10}$(AHF –like antigen)	No.	$\log_{10}$(AHF activity)	$\log_{10}$(AHF –like antigen)
1	-0.23	-0.3	1	-0.45	0.015
2	-0.18	-0.3	2	-0.43	-0.095
3	-0.13	-0.3	3	-0.42	-0.12
4	-0.16	-0.24	4	-0.41	-0.25
5	-0.025	-0.2	5	-0.38	-0.28
6	-0.12	-0.08	6	-0.35	-0.015
7	-0.075	-0.14	7	-0.34	0.1
8	-0.02	-0.15	8	-0.33	-0.13
9	-0.13	-0.05	9	-0.24	0.28
10	-0.08	-0.055	10	-0.24	0.15
11	-0.025	-0.09	11	-0.26	0.08
12	-0.06	-0.04	12	-0.26	-0.075
13	0	-0.08	13	-0.25	-0.04
14	0.05	-0.08	14	-0.22	-0.015
15	0.07	-0.1	15	-0.22	0.024
16	0.03	0.11	16	-0.21	-0.04
17	0.05	0	17	-0.175	-0.09
18	0.04	-0.03	18	-0.2	0.25
19	0.1	0	19	-0.19	0.175
20	0.075	0.02	20	-0.075	0.17
21	0.055	0.05	21	-0.015	0.15
22	0.06	0.1	22	-0.03	0.0135
23	0.09	0.09	23	-0.025	0.08
24	0.1	0.05			
25	0.11	0.035			
26	0.1	0.125			
27	0.12	0.125			
28	0.14	0.07			
29	0.21	0.11			

The first group of $n_1 = 29$ women were selected from a population of women who did not carry the hemophilia gene. This group was called the *normal* group. The second group of $n_2 = 23$ women was selected from known hemophilia A carriers (daughters of hemophiliacs, mothers with more than one hemophilic son, and mothers with one hemophilic son and other hemophilic relatives). This group was called the *obligatory carriers.* The pairs of observations (x_1, x_2) for the two groups are plotted on Figure 3. Also estimated contours are shown containing 50% and 95% of the probability for bivariate normal distributions centred at $\bar{\mathbf{x}}_1$ and $\bar{\mathbf{x}}_2$, respectively. Their common covariance matrix was taken as the pooled sample covariance matrix $\mathbf{S}_{\text{pooled}}$. In this example, bivariate normal distributions seem to fit the data fairly well.

The following information is given below:

$$\bar{\mathbf{x}}_1 = \begin{matrix} 0.00569 \\ -0.04655 \end{matrix}, \quad \bar{\mathbf{x}}_2 = \begin{matrix} -0.24870 \\ 0.01467 \end{matrix} \quad \text{and} \quad \mathbf{S}_{\text{pooled}}^{-1} = \begin{matrix} 121.6071 & -75.6327 \\ -75.6327 & 104.7043 \end{matrix}. \tag{39}$$

It follows from (6) that the optimal transformation vector is

$$\mathbf{u}^* = \mathbf{S}_{\text{pooled}}^{-1}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) = \begin{matrix} 121.6071 & -75.6327 \\ -75.6327 & 104.7043 \end{matrix} \begin{matrix} 0.2544 \\ -0.0612 \end{matrix} = \begin{matrix} 35.5657 \\ -25.6504 \end{matrix} \tag{40}$$

with

$$\bar{y}_1 = (\mathbf{u}^*)'\bar{\mathbf{x}}_1 = [35.5657 \quad -25.6504] \begin{matrix} 0.00569 \\ -0.04655 \end{matrix} = 1.3964 \tag{41}$$

and

$$\bar{y}_2 = (\mathbf{u}^*)'\bar{\mathbf{x}}_2 = [35.5657 \quad -25.6504] \begin{matrix} -0.24870 \\ -0.01467 \end{matrix} = -9.2214, \tag{42}$$

where the midpoint between these means is

$$\frac{\bar{y}_1 + \bar{y}_2}{2} = \frac{1.3964 - 9.2214}{2} = -3.9125. \tag{43}$$

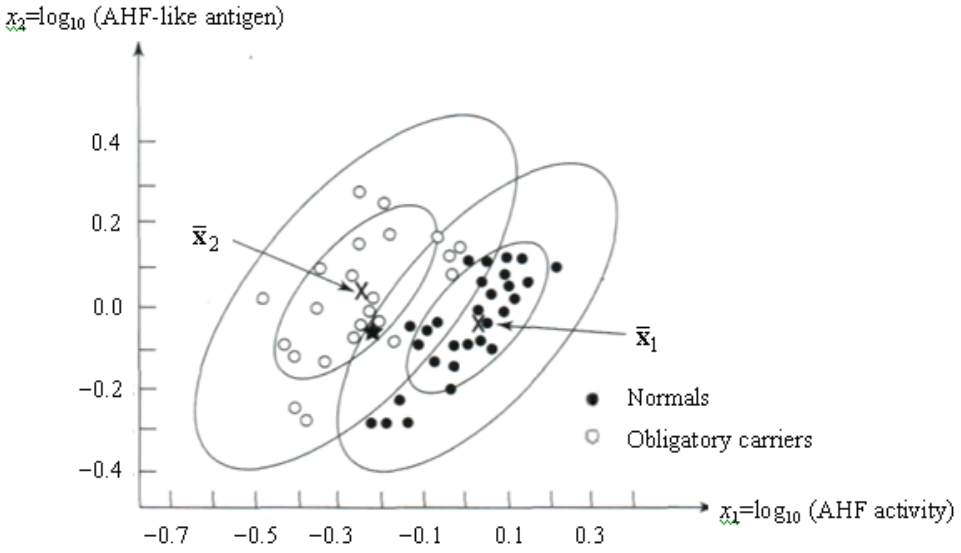

Figure 3. Scatter plots of [log$_{10}$ (AHF activity), log$_{10}$ (AHF-like antigen)] for the normal group and obligatory hemophilia A carriers

For instance, measurements of AHF activity and AHF-like antigen on a woman who may be a hemophilia A carrier give $x_1 = -0.210$ and $x_2 = -0.044$. Should this woman be classified as π_1 (normal) or π_2 (obligatory carrier)?

Using Fisher's approach, we obtain

$$y_0 = (\mathbf{u}^*)'\mathbf{x}_0 = [35.5657 \quad -25.6504] \begin{matrix} -0.210 \\ -0.044 \end{matrix} = -6.34 < \frac{\bar{y}_1 + \bar{y}_2}{2} = -3.9125. \tag{44}$$

Using the proposed approach based on the ratio of the maximum separations in the input data samples, we obtain

$$\frac{\max\limits_{\mathbf{u}\neq\mathbf{0}} \dfrac{[\mathbf{u}'(\bar{\mathbf{x}}_{1(0)} - \bar{\mathbf{x}}_2)]^2}{\mathbf{u}'\mathbf{S}_{\text{pooled}(1)}\mathbf{u}}}{\max\limits_{\mathbf{u}\neq\mathbf{0}} \dfrac{[\mathbf{u}'(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_{2(0)})]^2}{\mathbf{u}'\mathbf{S}_{\text{pooled}(2)}\mathbf{u}}} = \frac{(\bar{\mathbf{x}}_{1(0)} - \bar{\mathbf{x}}_2)'\mathbf{S}^{-1}_{\text{pooled}(1)}(\bar{\mathbf{x}}_{1(0)} - \bar{\mathbf{x}}_2)}{(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_{2(0)})'\mathbf{S}^{-1}_{\text{pooled}(2)}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_{2(0)})} = \frac{\bar{y}_{1(0)} - \bar{y}_2}{\bar{y}_1 - \bar{y}_{2(0)}} = \frac{1.029 + 8.84}{1.4 + 9.782} = 0.88 < 1. \tag{45}$$

Applying either (42) or (43), we classify the woman as π_2, an obligatory carrier. Thus, Fisher's approach and the proposed one give the same result in the above case.

4.2. Example 2

Consider the observations on $p = 2$ variables from $m = 3$ populations. The input data samples are given below.

π_1 ($n_1 = 3$) π_2 ($n_2 = 3$) π_3 ($n_3 = 3$)

$$\mathbf{x}^{n_1} = \begin{matrix} -2 & 5 \\ 0 & 3 \\ -1 & 1 \end{matrix}; \quad \mathbf{x}^{n_2} = \begin{matrix} 0 & 6 \\ 2 & 4 \\ 1 & 2 \end{matrix}; \quad \mathbf{x}^{n_3} = \begin{matrix} 1 & -2 \\ 0 & 0 \\ -1 & -4 \end{matrix}. \tag{46}$$

We found that

$$\bar{\mathbf{x}}_1 = \begin{matrix} -1 \\ 3 \end{matrix}; \quad \bar{\mathbf{x}}_2 = \begin{matrix} 1 \\ 4 \end{matrix}; \quad \bar{\mathbf{x}}_3 = \begin{matrix} 0 \\ -2 \end{matrix}; \quad \bar{\mathbf{x}} = \begin{matrix} 0 \\ 5/3 \end{matrix}; \tag{47}$$

$$\mathbf{B} = \sum_{i=1}^{3} 3(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' = \begin{matrix} 6 & 3 \\ 3 & 62 \end{matrix}; \tag{48}$$

$$\mathbf{W} = \sum_{i=1}^{3}\sum_{j=1}^{n_i}(\mathbf{x}_{ij} - \bar{\mathbf{x}}_i)(\mathbf{x}_{ij} - \bar{\mathbf{x}}_i)' = \sum_{i=1}^{3} n_i - 3\ \mathbf{S}_{\text{pooled}} = \begin{matrix} 6 & -2 \\ -2 & 24 \end{matrix}; \tag{49}$$

$$\mathbf{W}^{-1} = \begin{matrix} 0.171429 & 0.014286 \\ 0.014286 & 0.042857 \end{matrix}; \quad \mathbf{W}^{-1}\mathbf{B} = \begin{matrix} 1.071429 & 1.4 \\ 0.214286 & 2.7 \end{matrix}. \tag{50}$$

To solve for the $r \le \min(m-1, p) = \min(2, 2) = 2$ nonzero eigenvalues of $\mathbf{W}^{-1}\mathbf{B}$, we must solve

$$\left|\mathbf{W}^{-1}\mathbf{B} - \lambda\mathbf{I}\right| = \begin{vmatrix} 1.071429 - \lambda & 1.4 \\ 0.214286 & 2.7 - \lambda \end{vmatrix} = 0. \tag{51}$$

We find that $\lambda_1 = 2.867071$ and $\lambda_2 = 0.904357$. Then the corresponding normalized eigenvectors are obtained by solving

$$(\mathbf{W}^{-1}\mathbf{B} - \lambda_k\mathbf{I})\mathbf{u}_k^* = 0, \quad k = 1, 2, \tag{52}$$

and scaling the results such that

$$(\mathbf{u}_k^*)'\mathbf{S}_{\text{pooled}}\mathbf{u}_k^* = 1, \quad k = 1,2. \tag{53}$$

Thus, we obtain

$$\mathbf{u}_1^* = \begin{bmatrix} 0.386 & 0.495 \end{bmatrix}, \quad \mathbf{u}_2^* = \begin{bmatrix} 0.938 & -0.112 \end{bmatrix}. \tag{54}$$

Using Fisher's approach to classify the new observation $\mathbf{x}_0' = \begin{bmatrix} 1 & 3 \end{bmatrix}$ in accordance with (26), we have

$$\sum_{k=1}^{2}(y_{0k} - \bar{y}_{1k})^2 = \sum_{k=1}^{2}[(\mathbf{u}_k^*)'(\mathbf{x}_0 - \bar{\mathbf{x}}_1)]^2 = (1.871 - 1.099)^2 + (0.602 + 1.274)^2 = 4.11536, \tag{55}$$

$$\sum_{k=1}^{2}(y_{0k} - \bar{y}_{2k})^2 = \sum_{k=1}^{2}[(\mathbf{u}_k^*)'(\mathbf{x}_0 - \bar{\mathbf{x}}_2)]^2 = (1.871 - 2.366)^2 + (0.602 - 0.49)^2 = 0.257569, \tag{56}$$

$$\sum_{k=1}^{2}(y_{0k} - \bar{y}_{3k})^2 = \sum_{k=1}^{2}[(\mathbf{u}_k^*)'(\mathbf{x}_0 - \bar{\mathbf{x}}_3)]^2 = (1.871 + 0.99)^2 + (0.602 - 0.224)^2 = 8.328205. \tag{57}$$

Since the minimum of $\sum_{k=1}^{2}(y_{0k} - \bar{y}_{lk})^2$ occurs when $l = 2$, we allocate x_0 to population π_2. It will be noted that the approach proposed in this paper gives in the above case the same result. The situation, in terms of the classifiers $\bar{y}_i$ is illustrated on Figure 4.

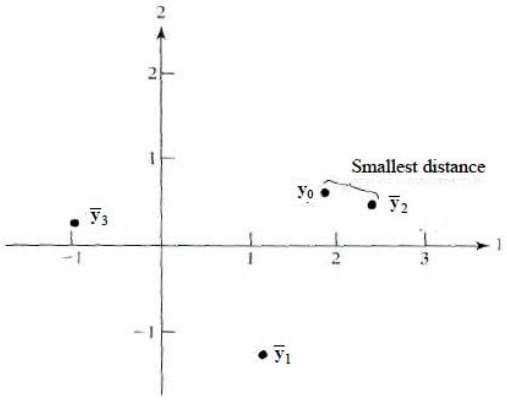

Figure 4. The points y_0, $\bar{y}_1$, $\bar{y}_2$, and $\bar{y}_3$ in the classification plane

5. Conclusions

The methodology described here can be extended in several different directions to handle various problems of pattern recognition that arise in practice (in particular, the problem of change-point detection in a sequence of multivariate observations).

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