Research on the local fractional Hilbert Transform based on fractal theory

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Abstract

With the development of computer technology and database technology, more and more MIS are implemented. In this paper, the local fractional Hilbert transform in fractal space is established. The characteristics of this local fractional transform are discussed in the following. Considering the basic properties of the local fractional Hilbert transforms, a kind of transform for local fractional is derived and analysed. The result shows that the transform can reach better performance.

Keywords: fractal space; local fractional Hilbert transform; local fractional derivative.

1 Introduction

The Hilbert transform arose in Hilbert's 1905 work on a problem posed by Riemann concerning analytic functions which has come to be known as the Riemann–Hilbert problem. Hilbert's work was mainly concerned with the Hilbert transform for functions defined on the circle. Some of his earlier work related to the Discrete Hilbert Transform dates back to lectures he gave in Gottingen. The results were later published by Hermann Weyl in his dissertation. Schur improved Hilbert's results about the discrete Hilbert transform and extended them to the integral case. These results were restricted to the spaces L2 and ℓ2. In 1928, Marcel Riesz proved that the Hilbert transform can be defined for u in Lp(R) for $1 \le p < \infty$, that the Hilbert transform is a bounded operator on Lp(R) for $1 < p < \infty$, and that similar results hold for the Hilbert transform on the circle as well as the discrete Hilbert transform. The Hilbert transform was a motivating example for Antoni Zygmund and Alberto Calderón during their study of singular integrals. Their investigations have played a fundamental role in modern harmonic analysis. Various generalizations of the Hilbert transform, such as the bilinear and tri-linear Hilbert transforms are still active areas of research today [1].

In mathematics and in signal processing, the Hilbert transform is a linear operator which takes a function, u(t), and produces a function, H(u)(t), with the same domain. The Hilbert transform is also important in the field of signal processing where it is used to derive the analytic representation of a signal u(t). This means that real signal u(t) is extended into the complex plane such that it satisfies the Cauchy-Riemann equations [2]. For example, the Hilbert transform leads to the harmonic conjugate of a given function in Fourier analysis, aka harmonic analysis. Equivalently, it is an example of a singular integral operator and of a Fourier multiplier.

The Hilbert transform was originally defined for periodic functions, or equivalently for functions on the circle, in which case it is given by convolution with the Hilbert kernel. More commonly, however, the Hilbert transform refers to a convolution with the Cauchy kernel, for functions defined on the real line R (the boundary of the upper half-plane). The Hilbert transform is closely related to the Paley–Wiener theorem, another result relating holomorphic functions in the upper half-plane and Fourier transforms of functions on the real line [3-4].

The Hilbert transform is named after David Hilbert, who first introduced the operator in order to solve a special case of the Riemann-Hilbert problem for holomorphic functions.

The function h with $h(t) = 1/(\pi t)$ is a non-causal filter and therefore cannot be implemented as is, if u is a time-dependent signal. If u is a function of a non-temporal variable (e.g., spatial) the non-causality might not be a problem. The filter is also of infinite support which may be a problem in certain applications. Another issue relates to what happens with the zero frequency (DC), which can be avoided by assuring that s does not contain a DC-component [5].

A practical implementation in many cases implies that a finite support filter, which in addition is made causal by means of a suitable delay, is used to approximate the computation. The approximation may also imply that only a specific frequency range is subject to the characteristic phase shift related to the Hilbert transform. It also can be seen from the quadrature filter [6].

2 The local fractional hilbert transform and some examples

In the past ten years, Local fractional calculus[7-8] has been widely applied to many fields such as mathematics , image processing and signal processing etc .Some authors have given many definitions of local fractional derivatives and local fractional integrals (also called fractal calculus) [9]. Hereby we rewrite the following local fractional derivative which is given by [10].

$$f^{(\alpha)}(x0) = \left.\frac{df(x)}{dx^{\alpha}}\right|_{x=x_0} = \lim_{\delta x \to 0} \frac{\Delta^{\alpha}(f(x) - f(x_0))}{(x-x_0)^{\alpha}} \quad \text{for} \quad 0 < a \le 1, \tag{1}$$

where

$$\Delta^{\alpha}(f(x) - f(x_0)) \cong \Gamma(1+\alpha) \lim_{x\to\infty} \Delta(f(x) - f(x_0)), \tag{2}$$

and local fractional integral of $f(x)$ defined by [11-12].

$$ {}_aI_b^{(\alpha)} f(t) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^{\alpha} = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t\to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^{\alpha} \tag{3}$$

With

$\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_1, \Delta t_2, \cdots, \Delta t_j, \cdots\}$,

where for $j = 1,2,\cdots,N-1$, $[t_j, t_{j+1}]$ is a partition of the interval $[a,b]$ and $t_0 = a$, $t_N = b$.

The aim of this paper is to define local fractional Hilbert transforms based on Local fractional calculus. This paper is organized as follows. In section 2, local fractional Hilbert transforms is defined. Section 3 presents Properties of local fractional Hilbert transforms.

In the section, both local fractional Hilbert transform and its inverse transform are defined.

Definition 2.1:(The local fractional Hilbert transform).

If $f(x)$ is defined on the real line $-\infty < x < \infty$, its local fractional Hilbert transform, denoted by $f_x^{H,\alpha}(x)$ is defined by

$$H_{\alpha}\{f(t)\} = \hat{f}_H^{\alpha}(x) = \frac{1}{\Gamma(1+\alpha)} \oint_R \frac{f(t)}{(t-x)^{\alpha}}(dt)^{\alpha}, \tag{4}$$

where x is real and the integral is treated as a Canchy principal value, that is,

$$\frac{1}{\Gamma(1+\alpha)} \oint_R \frac{f(t)}{(t-x)^{\alpha}}(dt)^{\alpha} = \lim_{\varepsilon\to 0}\left[\frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{x-\varepsilon} \frac{f(t)}{(t-x)^{\alpha}}(dt)^{\alpha} + \frac{1}{\Gamma(1+\alpha)} \int_{x+\varepsilon}^{\infty} \frac{f(t)}{(t-x)^{\alpha}}(dt)^{\alpha}\right]. \tag{5}$$

To obtain the inverse local fractional Hilbert transform, write again (4) as

$$\hat{f}_H^{\alpha}(x) = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \frac{f(t)}{(t-x)^{\alpha}}(dt)^{\alpha} = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) g(x-t)(dt)^{\alpha} = f(x) * g(x), \tag{6}$$

where $g(x) = -\frac{1}{x^{\alpha}}$, application of the local fractional Yang-Fourier transform [10-11] with respect to x gives

$$f_{\omega}^{F,\alpha}(\omega) = \frac{f_{H,\omega}^{F,\alpha}(\omega)}{g_{\omega}^{F,\alpha}(\omega)}, g_{\omega}^{F,\alpha}(\omega) = i^{\alpha}\pi^{\alpha} \operatorname{sgn}_{\alpha} \omega^{\alpha}. \tag{7}$$

Taking the inverse Yang-Fourier transform [12], we find the solution for $f(x)$ as

$$f(x) = \frac{1}{(2\pi)^{\alpha}} \int_{-\infty}^{\infty} i^{\alpha}\pi^{\alpha} \operatorname{sgn}_{\alpha} \omega^{\alpha} f_{H,\omega}^{F,\alpha}(\omega) E_{\alpha}(i^{\alpha}\omega^{\alpha}x^{\alpha})(d\omega)^{\alpha}. \tag{8}$$

This is by the Convolution Theorem.

$$f(x) = \frac{1}{(2\pi)^{\alpha}} \oint_R \frac{\hat{f}_H^{\alpha}(\xi)}{(x-\xi)^{\alpha}}(d\xi)^{\alpha} = -\frac{1}{(2\pi)^{\alpha}} \oint_R \frac{\hat{f}_H^{\alpha}(\xi)}{(x-\xi)^{\alpha}}(d\xi)^{\alpha} = -H\left\{\hat{f}_H^{\alpha}(\xi)\right\}. \tag{9}$$

Obviously, $-H_{\alpha}^2\{f(t)\} = -H_{\alpha}[H_{\alpha}\{f(t)\}] = f(x)$ and hence, $H_{\alpha}^{-1} = -H_{\alpha}$. Thus, the inverse local fractional Hilbert transform is given by

$$f(t) = H_{\alpha}^{-1}\left\{\hat{f}_H^{\alpha}(x)\right\} = -H_{\alpha}\left\{\hat{f}_H^{\alpha}(x)\right\} = -\frac{1}{(2\pi)^{\alpha}} \oint_R \frac{\hat{f}_H^{\alpha}(\xi)}{(x-\xi)^{\alpha}}(d\xi)^{\alpha}. \tag{10}$$

Example 2.1:

Find the local fractional Hilbert transform of

$$f(t) = \frac{t^{\alpha}}{t^{2\alpha} + a^{2\alpha}}, \quad a > 0$$

We get, by definition,

$$\hat{f}_H^{\alpha}(x) = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \frac{t^{\alpha}}{(t^{2\alpha}+a^{2\alpha})(t-x)^{\alpha}}(dt)^{\alpha} = \frac{1}{(a^{2\alpha}+x^{2\alpha})\Gamma(1+\alpha)} \int_{-\infty}^{\infty}\left[\frac{a^{2\alpha}}{(t^{2\alpha}+a^{2\alpha})} + \frac{x^{\alpha}}{(t-x)^{\alpha}} - \frac{x^{\alpha}t^{\alpha}}{(t^{2\alpha}+a^{2\alpha})}\right](dt)^{\alpha}$$
$$= \frac{1}{(a^{2\alpha}+x^{2\alpha})\Gamma(1+\alpha)}\left[a^{2\alpha}\int_{-\infty}^{\infty} \frac{a^{2\alpha}}{(t^{2\alpha}+a^{2\alpha})}(dt)^{\alpha} + x^{\alpha}\frac{x^{\alpha}}{(t-x)^{\alpha}}(dt)^{\alpha} - x^{\alpha}\frac{x^{\alpha}t^{\alpha}}{(t^{2\alpha}+a^{2\alpha})}(dt)^{\alpha}\right]. \tag{11}$$

The second and third integrals as the Cauchy principal value vanish and hence, only the first integral makes a non-zero contribution.

Thus, we have

$$\hat{f}_H^{\alpha}(x) = \frac{a^{2\alpha}}{(a^{2\alpha}+x^{2\alpha})} \arctan_{\alpha} \left.\frac{t^{\alpha}}{a^{\alpha}}\right|_{-\infty}^{\infty} = \frac{\pi^{\alpha}a^{\alpha}}{(a^{2\alpha}+x^{2\alpha})}. \tag{12}$$

Example 2.2: Find the local fractional Hilbert transform of $f(t)=\cos_\alpha \omega^\alpha t^\alpha$, $f(t)=\sin_\alpha \omega^\alpha t^\alpha$

$$\begin{aligned}\hat{f}_H^\alpha(x) &= \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{\cos_\alpha \omega^\alpha t^\alpha}{(t-x)^\alpha}(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{\cos_\alpha\left[\omega^\alpha (t-x)^\alpha+\omega^\alpha x^\alpha\right]}{(t-x)^\alpha}(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{\cos_\alpha \omega^\alpha (t-x)^\alpha \cos_\alpha \omega^\alpha x^\alpha}{(t-x)^\alpha}(dt)^\alpha - \int_{-\infty}^{\infty}\frac{\sin_\alpha \omega^\alpha (t-x)^\alpha \sin_\alpha \omega^\alpha x^\alpha}{(t-x)^\alpha}(dt)^\alpha \\ &= \cos_\alpha \omega^\alpha x^\alpha \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{\cos_\alpha \omega^\alpha (t-x)^\alpha}{(t-x)^\alpha}(dt)^\alpha - \sin_\alpha \omega^\alpha x^\alpha \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{\sin_\alpha \omega^\alpha (t-x)^\alpha}{(t-x)^\alpha}(dt)^\alpha .\end{aligned} \tag{13}$$

Which is, the new variable $T=t-x$,

$$\hat{f}_H^\alpha(x)=\cos_\alpha \omega^\alpha x^\alpha \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{\cos_\alpha \omega^\alpha T^\alpha}{T^\alpha}(dT)^\alpha - \sin_\alpha \omega^\alpha x^\alpha \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{\sin_\alpha \omega^\alpha T^\alpha}{T^\alpha}(dT)^\alpha \tag{14}$$

Obviously, the first integral vanishes because its integrand is an odd function of T. On the other hand, the second integral makes a non-zero contribution so that (2.8) gives

$$H_\alpha\left\{\cos_\alpha \omega^\alpha t^\alpha\right\}=-\pi^\alpha \sin_\alpha \omega^\alpha x^\alpha . \tag{15}$$

Similarly, it can be shown that

$$H_\alpha\left\{\sin_\alpha \omega^\alpha t^\alpha\right\}=-\pi^\alpha \cos_\alpha \omega^\alpha x^\alpha . \tag{16}$$

3 Basic properties of the local fractional hilbert transforms

Theorem 3.1: If $H_\alpha\{f(t)\}=\hat{f}_H^\alpha(x)$, then the following properties hold:

$$H_\alpha\{f(t+a)\}=\hat{f}_H^\alpha(x+a), \tag{17}$$

$$H_\alpha\{f(at)\}=\hat{f}_H^\alpha(ax), \tag{18}$$

$$H_\alpha\{f(-at)\}=-\hat{f}_H^\alpha(-ax), \tag{19}$$

$$H_\alpha\{f^\alpha(t)\}=\frac{d^\alpha \hat{f}_H^\alpha(x)}{dx^\alpha}, \tag{20}$$

$$H_\alpha\{tf(t)\}=x^\alpha \hat{f}_H^\alpha(x)+\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}f(t)(dt)^\alpha, \tag{21}$$

$$F_\alpha\{H_\alpha\{f(t)\}\}=-i^\alpha \operatorname{sgn}_\alpha \omega^\alpha F_\alpha\{f(x)\}, \tag{22}$$

Theorem 3.2: If f and $g\in L_{1,\alpha}(\mathbb{R})$ are such that their local fractional Hilbert transforms are also in $L_{1,\alpha}(\mathbb{R})$, Then

$$H_\alpha(f*g)(x)=(H_\alpha f*g)(x)=(f*H_\alpha g)(x). \tag{23}$$

A practical implementation in many cases implies that a finite support filter, which in addition is made causal by means of a suitable delay, is used to approximate the computation.

Proof: We obtain, by definition, $\xi=t-y$

$$\begin{aligned}H_\alpha(f*g)(x) &= \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{1}{(t-x)^\alpha}\left[\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}f(y)g(t-y)(dy)^\alpha\right](dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}f(y)(dy)^\alpha\left[\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{g(\xi)}{(\xi-(x-y))^\alpha}(d\xi)^\alpha\right]=\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}f(y)(H_\alpha g)(x-y)(dy)^\alpha=(f*H_\alpha g)(x).\end{aligned} \tag{24}$$

Likewise, we can obtain the second result in equation (23).

4 Discrete hilbert transform

For a discrete function, u[n], with discrete-time Fourier transform (DTFT), u(ω), the Hilbert transform is given by:

$$H(u)[n]=DTFT^{-1}\{U(\omega)\bullet\sigma_H(\omega)\}, \tag{25}$$

where:

$$\sigma_H(\omega)\overset{def}{=}\begin{cases}e^{+i\pi/2}, & -\pi<\omega<0\\ e^{-i\pi/2}, & 0<\omega<\pi\\ 0, & \omega=-\pi,0,\pi\end{cases}. \tag{26}$$

And by the convolution theorem, an equivalent formulation is: $H(u)[n]=u[n]*h[n]$, (27)

where:

$$h[n]\overset{def}{=}DTFT^{-1}\{\sigma_H(\omega)\}=\begin{cases}\frac{2}{\pi n}, & for\ \ n\ \ odd\\ 0, & for\ \ n\ \ even\end{cases} \tag{28}$$

When the convolution is performed numerically, an FIR approximation is substituted for h[n], and we see roll off of the pass band at the low and high ends (0 and Nyquist), resulting in a band pass filter. The high end can be restored, by an FIR that more closely resembles samples of the smooth, continuous-time h(t). But as a practical matter, a properly-sampled u[n] sequence has no useful components at those frequencies. As the impulse response gets longer, the low end frequencies are also restored [12].

With an FIR approximation to h[n], a method called overlap-save is an efficient way to perform the convolution on a long u[n] sequence. Sometimes the array FFT {h[n]} is replaced by corresponding samples of $\sigma_H(\omega)$. That has the effect of convolving with the periodic summation:

$$h_N[n]\overset{def}{=}\sum_{m=-\infty}^{\infty}h[n-mN]. \tag{29}$$

Figure 1 compares a half-cycle of $h_N[n]$ with an equivalent length portion of h[n]. The difference between them and the fact that they are not shorter than the segment length (N) are sources of distortion that are managed (reduced) by increasing the segment length and overlap parameters.

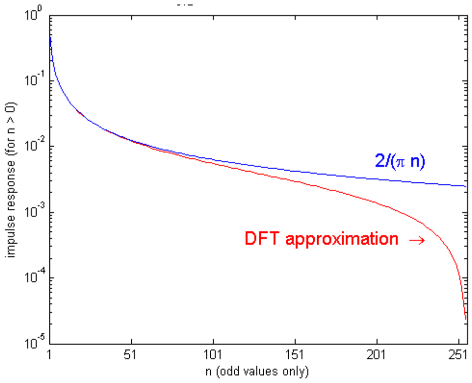

FIGURE 1 The comparison a half-cycle of $h_N[n]$ with an equivalent length portion of h[n]

The popular MATLAB function, Hilbert (u, N), returns an approximate discrete Hilbert transform of u[n] in the imaginary part of the complex output sequence. The real part is the original input sequence, so that the complex output is an analytic representation of u[n]. Similar to the discussion above, Hilbert (u, N) only uses samples of the sgn(ω) distribution and therefore convolves with $h_N[n]$. Distortion can be managed by choosing N larger than the actual u[n] sequence and discarding an appropriate number of output samples. An example of this type of distortion is shown in figure 2.

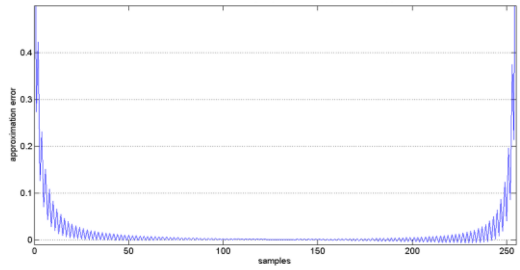

FIGURE 2 The difference between sin(wt) and an approximate

5 Conclusions

In present paper we give the local fractional Hilbert transform as follows:

$$H_\alpha\{f(t)\} = \hat{f}_H^\alpha(x) = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{f(t)}{(t-x)^\alpha}(dt)^\alpha . \qquad (30)$$

$for \quad 0<\alpha\leq 1$

And its inverse transform

$$f(t) = H_\alpha^{-1}\left\{\hat{f}_H^\alpha(x)\right\} = -H_\alpha\left\{\hat{f}_H^\alpha(x)\right\} = -\frac{1}{(2\pi)^\alpha}\frac{\hat{f}_H^\alpha(\xi)}{(x-\xi)^\alpha}(d\xi)^\alpha . \qquad (31)$$

The transforming functions are local fractional continuous. That is to say, it is fractal function defined on fractal sets. Hilbert transforms in integer space are the special case of fractal dimension $\alpha=1$. It is a tool to deal with differential equation with local fractional derivative.

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