Approximation of unit-hypercubic infinite two-sided noncooperative game via dimension-dependent irregular samplings and reshaping the multidimensional payoff matrices into flat matrices for solving the corresponding bimatrix game

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Received 1 May 2015, www.cmnt.lv

Abstract

The problem of solving unit-hypercubic infinite two-sided noncooperative games is considered. The ultimate goal is to approximate the infinite game with bimatrix game, ranking the approximation accurateness. This is fulfilled in three stages. Primarily the players' payoff functions are sampled under stated conditions of dimension-dependent irregular samplings. Then the sampled payoff functions as multidimensional payoff matrices are mapped into ordinary flat matrices under a reversible matrix map. Finally, after obtaining the solution of the corresponding bimatrix game, equilibrium finite support strategies are checked out for their consistency, being used as the approximation accurateness rank. If consistent, then the bimatrix game can be regarded as the approximation of the initial noncooperative game. For particular cases, conditions of the weakened consistency are stipulated. Different types of consistency ensure the corresponding bimatrix game solution varying reasonably by changing the sampling steps minimally. If the solution is not even weakly consistent by the most primitive consistency in ranking the approximation accurateness, then the sampling intervals should be shortened. If any shortening is impossible then the sampling points must be set otherwise. The suggested approximation tool is fully applicable to games, which are isomorphic to the unit-hypercubic infinite two-sided noncooperative game.

Keywords: two-sided noncooperative games, unit hypercube, approximation, irregular sampling, bimatrix game, multidimensional matrix, equilibrium finite support strategy, approximate solution, equilibrium solution consistency

1 Games in Euclidean finite-dimensional subspaces of non-single dimension

Conflict events are result of everlasting natural disproportion of resources and demands. To allocate resources properly and adequately, there are game models whose principal purpose is in ensuring equilibrium, equity, efficiency, and utility of the allocation [1, 2]. Two-sided noncooperative game (TSNCG) solutions are applied to economics [3, 4], politics [5], military science, jurisprudence [6, 7], social [7, 8] and ecologic processes [9, 10], technological and technical processes, reducing their risks or losses on average [1, 11, 12]. However, often there is a problem of uncertainty of the equilibrium type [13]. Nowadays, there are a lot of equilibrium types, whose origins are regarding mainly to Nash and Pareto equilibrium [1, 2, 4, 8, 13]. Another problem is infiniteness and dimensions. The matter is that for a great many of conflict events the player's pure strategy is an action, featured with a sequence of parameters, belonging to some intervals of their acceptable values. Thus players get infinite multidimensional sets of their pure strategies. These sets are equivalent to Euclidean finite-dimensional subspaces of non-single dimension [14, 15]. And even TSNCG on such subspaces' product is very difficult to get solved analytically [1, 2, 16], unless the players' payoff functions (PPF) are specific cases.

2 Solutions of infinite TSNCG

There is a few ways for obtaining the exact solution of infinite TSNCG. When the game is infinite, there is no any universal method of solving, but just narrowly specified technique, oriented on particular cases [1, 17]. One of those particularities works on compact games with continuous PPF, having solutions at least in mixed strategies [1, 2, 17, 18]. In more general, bounded games can be solved in ε-equilibrium finite support strategies (FSS) [1, 19, 20]. And with unbounded or non-measurable or discontinuous PPF there are only approximations available.

The approximation implies either of two directions: straightforward approximation over infiniteness or initial transition from infiniteness to finite game. Every direction has its own shortcomings. While approximating straightforwardly over infiniteness, one should be aware of hard analytical reasonings, including limit theorems not always giving the constructive decision even if proved. Moreover, the solution with a mixed infinite support strategy, whatever it is (exact or approximate), carries a problem of its implementation in practice. Impossibility of full practicability of the infinite support is from that the number of plays (rounds of game or its recurrence) is finite. This apparent lack of infinite approximation is beyond finite games [21, 22]. Transition from infiniteness to finite game will definitely give FSS, whose practical implementation is easier. But the transition must be fulfilled carefully, not losing important properties of PPF. Rank of this carefulness is to be ascertained later. When ε- net-construction technique is applied, the rank is roughly equal to ε. Well, ε_1-equilibrium FSS are more accurate than ε_2 -equilibrium FSS for $\varepsilon_1 < \varepsilon_2$, although, firstly, it is unknown how to select ε_1. Secondly, it is also unknown whether exists a limit $\varepsilon_0 < \varepsilon_1$ such that ε-equilibrium FSS have similar accurateness as ε_0 -equilibrium FSS for $\varepsilon < \varepsilon_0$.

Finite TSNCG is represented with two multidimensional matrices (MDM), whose elements are values of the sampled PPF. If every player acts within single dimension, its MDM is ordinal flat two-dimensional array. Then finite TSNCG is bimatrix game (BMG), which is solved with well-known methods of linear programming by algorithm of Lemke — Howson [23] or related simplex-pivoting-operation modifications based on this algorithm [24, 25]. Nonetheless, if MDM is not a flat matrix, the corresponding finite TSNCG cannot be solved as BMG — the additional transformation of MDM into two-dimensional array is needed.

3 Goal and tasks

In suggesting an approximation method, a class of infinite TSNCG should be considered. Other game classes, if they are isomorphic to the considered class, will be treated similarly. For instance, if players' action spaces are compacts in Euclidean finite-dimensional spaces then they can be "normalized" to unit hypercubes in these spaces. Then, without loss of generality, the class of infinite TSNCG on unit hypercube is going to be considered.

The ultimate goal is to approximate unit-hypercubic infinite TSNCG with BMG, exposing accurateness of the approximation and defining its eligibility. This goal is going to be attained via accomplishing four tasks. Primarily the conditions of acceptance of the sampled PPF must be declared. They are for transforming PPF into MDM and not losing important properties of PPF. Then, having represented the finite TSNCG with two MDM, there must be substantiated a mapping of MDM into ordinary flat matrices. This mapping will allow to solve the corresponding BMG and to map its solution to the initial finite TSNCG. The task at the third stage is to rank accurateness of the approximation for elementary case. Eventually, the conception of this rank has to be spread out to more general cases. If the solution of BMG is of the satisfactory accurateness rank then this solution is going to be called the approximation of the initial TSNCG solution. The rank ought to answer whether BMG solution varies vastly by changing the sampling steps. And these steps are to be selected regarding the dimension, where along the dimension the sampling interval can vary as needed.

4 Conditions of sampling PPF

Let there be a TSNCG

$$\left\langle H_1, H_2, K_1(\mathbf{X}, \mathbf{Y}), K_2(\mathbf{X}, \mathbf{Y})\right\rangle \tag{1}$$

with the players' pure strategies sets

$$H_1 = \bigtimes_{m=1}^{M} [0; 1] \subset \mathbb{R}^M, \ M \in \mathbb{N} \tag{2}$$

and

$$H_2 = \bigtimes_{n=1}^{N} [0; 1] \subset \mathbb{R}^N, \ N \in \mathbb{N} \tag{3}$$

and PPF $K_1(\mathbf{X}, \mathbf{Y})$ and $K_2(\mathbf{X}, \mathbf{Y})$, defined on $(M+N)$-dimensional unit hypercube

$$H_1 \times H_2 = \left\{ \bigtimes_{m=1}^{M} [0; 1] \right\} \times \left\{ \bigtimes_{n=1}^{N} [0; 1] \right\} = \left\{ \bigtimes_{k=1}^{M+N} [0; 1] \right\} \subset \mathbb{R}^{M+N} \tag{4}$$

by

$$\mathbf{X} = [x_m]_{1\times M} \in H_1, \tag{5}$$

$$\mathbf{Y} = [y_n]_{1\times N} \in H_2. \tag{6}$$

Primarily, TSNCG is assumed to be such that each of the functions $K_1(\mathbf{X}, \mathbf{Y})$ and $K_2(\mathbf{X}, \mathbf{Y})$ is differentiable with respect to any of variables

$$\left\{ \{x_m\}_{m=1}^{M}, \{y_n\}_{n=1}^{N} \right\}. \tag{7}$$

Also let there exist mixed derivatives of each of those functions by any combination of variables (7) in any situation

$$\{\mathbf{X}, \mathbf{Y}\} \in H_1 \times H_2,$$

where every variable is included no more than just once. Afterwards, these conditions can be broken.

PPF $\{K_r(\mathbf{X}, \mathbf{Y})\}_{r\in\{1, 2\}}$ are sampled along each of dimensions of hypercube (4) with a specific sampling rule. These are going to be dimension-dependent irregular samplings, where each dimension is the unit segment. Let $S_m^{\langle 1\rangle}$ be the number of intervals between the selected points in m-th dimension of hypercube (2), and $S_n^{\langle 2\rangle}$ be the number of intervals between the selected points in n-th dimension of hypercube (3). In the utmost case of sampling, $S_m^{\langle 1\rangle} \in \mathbb{N}$ and $S_n^{\langle 2\rangle} \in \mathbb{N}$. Therefore, endpoints of the unit segment are included into the sampling necessarily, while there is no fixed sampling step. Thus, in m-th dimension the first player (FP) instead of the segment $[0; 1]$ of values of m-th component of its pure strategy (5) now possesses the set of points

$$D_m^{\langle 1\rangle}\left(S_m^{\langle 1\rangle}\right) = \left\{ x_m^{\langle s_m\rangle} \right\}_{s_m=1}^{S_m^{\langle 1\rangle}+1}, \ x_m^{\langle 1\rangle} = 0,$$
$$x_m^{\left\langle S_m^{\langle 1\rangle}+1\right\rangle} = 1, \tag{8}$$
$$x_m^{\langle d_m\rangle} < x_m^{\langle d_m+1\rangle}, \ \forall\, d_m = \overline{1, S_m^{\langle 1\rangle}}, \ m = \overline{1, M}.$$

In n-th dimension the second player (SP) instead of the segment $[0; 1]$ of values of n-th component of its pure strategy (6) now possesses the set of points

$$D_n^{\langle 2\rangle}\left(S_n^{\langle 2\rangle}\right) = \left\{ y_n^{\langle s_n\rangle} \right\}_{s_n=1}^{S_n^{\langle 2\rangle}+1}, \ y_n^{\langle 1\rangle} = 0,$$
$$y_n^{\left\langle S_n^{\langle 2\rangle}+1\right\rangle} = 1, \tag{9}$$
$$y_n^{\langle d_n\rangle} < y_n^{\langle d_n+1\rangle}, \ \forall\, d_n = \overline{1, S_n^{\langle 2\rangle}}, \ n = \overline{1, N}.$$

Subsequently, the finite hypercubic irregular lattice (FHCIL)

$$D^{\langle 1\rangle} = \bigtimes_{m=1}^{M} D_m^{\langle 1\rangle}\left(S_m^{\langle 1\rangle}\right) = \bigtimes_{m=1}^{M} \left\{\left\{x_m^{\langle s_m\rangle}\right\}_{s_m=1}^{S_m^{\langle 1\rangle}+1}\right\} \tag{10}$$

substitutes the hypercube (2), and FHCIL

$$D^{\langle 2\rangle} = \bigtimes_{n=1}^{N} D_n^{\langle 2\rangle}\left(S_n^{\langle 2\rangle}\right) = \bigtimes_{n=1}^{N} \left\{\left\{y_n^{\langle s_n\rangle}\right\}_{s_n=1}^{S_n^{\langle 2\rangle}+1}\right\} \tag{11}$$

substitutes the hypercube (3). This transforms the infinite TSNCG (1) to finite TSNCG

$$\left\langle D^{\langle 1\rangle},\, D^{\langle 2\rangle},\, K_1(\mathbf{X},\,\mathbf{Y}),\, K_2(\mathbf{X},\,\mathbf{Y})\right\rangle,\ \mathbf{X}\in D^{\langle 1\rangle},\ \mathbf{Y}\in D^{\langle 2\rangle} \tag{12}$$

on FHCIL $D^{\langle 1\rangle}\times D^{\langle 2\rangle}$, where hypersurface $K_r(\mathbf{X},\,\mathbf{Y})$ is transformed into $(M+N)$-dimensional array (matrix)

$$\left\{\left\{K_r(\mathbf{X},\,\mathbf{Y})\right\}_{\mathbf{X}\in D^{\langle 1\rangle}}\right\}_{\mathbf{Y}\in D^{\langle 2\rangle}},\ r\in\{1,\,2\}. \tag{13}$$

The sampling numbers

$$\left\{\left\{S_m^{\langle 1\rangle}\right\}_{m=1}^{M},\ \left\{S_n^{\langle 2\rangle}\right\}_{n=1}^{N}\right\} \tag{14}$$

shall clearly not be assigned arbitrarily, because the sampling mustn't erase specificities of PPF. These specificities are information about local extremums and gradient over hypersurfaces $\left\{K_r(\mathbf{X},\,\mathbf{Y})\right\}_{r\in\{1,\,2\}}$. That is $\forall\ s_m=\overline{1,\,S_m^{\langle 1\rangle}}$ and $\forall\ s_n=\overline{1,\,S_n^{\langle 2\rangle}}$ by $m=\overline{1,\,M}$, $n=\overline{1,\,N}$, $r\in\{1,\,2\}$ there ought to be

$$\begin{gathered}\frac{\partial^{M+N}K_r(X,Y)}{\partial x_1\partial x_2\ldots\partial x_M\,\partial y_1\partial y_2\ldots\partial y_N}\geq 0,\\ \frac{\partial^{M+N}K_r(X,Y)}{\partial x_1\partial x_2\ldots\partial x_M\,\partial y_1\partial y_2\ldots\partial y_N}\leq 0,\\ \forall\ x_m\in\left[x_m^{\langle s_m\rangle};\ x_m^{\langle s_m+1\rangle}\right],\ \forall\ y_n\in\left[y_n^{\langle s_n\rangle};\ y_n^{\langle s_n+1\rangle}\right].\end{gathered} \tag{15}$$

and

$$\begin{gathered}\left|\frac{\partial^{M+N}K_r(X,Y)}{\partial x_1\partial x_2\ldots\partial x_M\,\partial y_1\partial y_2\ldots\partial y_N}\right|\leq\gamma,\\ \forall\ x_m\in\left[x_m^{\langle s_m\rangle};\ x_m^{\langle s_m+1\rangle}\right],\ \forall\ y_n\in\left[y_n^{\langle s_n\rangle};\ y_n^{\langle s_n+1\rangle}\right].\end{gathered} \tag{16}$$

for some $\gamma>0$, implying tolerable fluctuations of the hypersurface $K_r(\mathbf{X},\,\mathbf{Y})$ on every one of segments

$$\left\{\left\{\left\{\left[x_m^{\langle s_m\rangle};\ x_m^{\langle s_m+1\rangle}\right]\right\}_{s_m=1}^{S_m^{\langle 1\rangle}}\right\}_{m=1}^{M},\ \left\{\left\{\left[y_n^{\langle s_n\rangle};\ y_n^{\langle s_n+1\rangle}\right]\right\}_{s_n=1}^{S_n^{\langle 2\rangle}}\right\}_{n=1}^{N}\right\}.$$

The following assertion directs for choosing the numbers (14) and points

$$\left\{\left\{\left\{x_m^{\langle s_m\rangle}\right\}_{s_m=2}^{S_m^{\langle 1\rangle}}\right\}_{m=1}^{M},\ \left\{\left\{y_n^{\langle s_n\rangle}\right\}_{s_n=2}^{S_n^{\langle 2\rangle}}\right\}_{n=1}^{N}\right\} \tag{17}$$

in order to sample PPF under conditions (15) and (16) from the hypercube (4) down to FHCIL $D^{\langle 1\rangle}\times D^{\langle 2\rangle}$.

Theorem 1. If local extremums of hypersurfaces $K_1(\mathbf{X},\,\mathbf{Y})$ and $K_2(\mathbf{X},\,\mathbf{Y})$ are reached at points having only components (17) then, if inequalities (16) hold $\forall\ s_m=\overline{1,\,S_m^{\langle 1\rangle}}$ and $\forall\ s_n=\overline{1,\,S_n^{\langle 2\rangle}}$ by $m=\overline{1,\,M}$, $n=\overline{1,\,N}$, $r\in\{1,\,2\}$, PPF $\left\{K_r(\mathbf{X},\,\mathbf{Y})\right\}_{r\in\{1,\,2\}}$ are sampled with (8)-(11).

Proof. Since having local extremums only with components (17), neither the hypersurface $K_1(\mathbf{X},\,\mathbf{Y})$ nor the hypersurface $K_2(\mathbf{X},\,\mathbf{Y})$ have local extremums on every one of intervals

$$\left\{\left\{\left\{\left(x_m^{\langle s_m\rangle};\ x_m^{\langle s_m+1\rangle}\right)\right\}_{s_m=1}^{S_m^{\langle 1\rangle}}\right\}_{m=1}^{M},\ \left\{\left\{\left(y_n^{\langle s_n\rangle};\ y_n^{\langle s_n+1\rangle}\right)\right\}_{s_n=1}^{S_n^{\langle 2\rangle}}\right\}_{n=1}^{N}\right\}.$$

Hence, for the differentiable PPF $\left\{K_r(\mathbf{X},\,\mathbf{Y})\right\}_{r\in\{1,\,2\}}$, conditions (15) hold as well. The theorem has been proved.

For conditions (16), parameter γ is pre-assigned on reasoning about the value

$$\gamma\leqslant\lambda\cdot\left(\max_{r\in\{1,\,2\}}\max_{\mathbf{X}\in H_1}\max_{\mathbf{Y}\in H_2}K_r(\mathbf{X},\,\mathbf{Y})-\min_{r\in\{1,\,2\}}\min_{\mathbf{X}\in H_1}\min_{\mathbf{Y}\in H_2}K_r(\mathbf{X},\,\mathbf{Y})\right) \tag{18}$$

by, say,

$$\lambda\in\{0.001,\ 0.005,\ 0.01,\ 0.02\}$$

or other practically appropriate values for the scale factor λ. Nevertheless, some conditions below may cause need to resample PPF with the lowered parameter γ. These conditions, telling whether the approximate solution of TSNCG (1) is stable enough, are going to be applied to the finite TSNCG (12) solution. Before stating them, MDM (13) should be represented as ordinary flat matrices, letting solve the finite TSNCG (12) as BMG.

5 Mapping PPF as MDM into ordinary flat matrices

The quadruple (12) is a finite TSNCG but it is not BMG unless $M=N=1$. Therefore, mapping PPF as MDM (13) into ordinary flat matrices will allow to find the finite game solution with any acceptable methods for solving BMG, including linear programming methods on the basis of algorithm of Lemke — Howson [23] and its modifications [24, 25]. Denote MDM (13) as $(M+N)$-dimensional matrix $P_r(0)=\left[p_J^{\langle r\rangle}\right]_F$ of the format

$$F=\left\{\bigtimes_{m=1}^{M}\left(S_m^{\langle 1\rangle}+1\right)\right\}\times\left\{\bigtimes_{n=1}^{N}\left(S_n^{\langle 2\rangle}+1\right)\right\}, \tag{19}$$

whose $(M+N)$-position indices

$$J = \{ j_k \}_{k=1}^{M+N}, \; j_m \in \left\{ \overline{1, \, S_m^{\langle 1 \rangle} + 1} \right\}, \; j_{M+n} \in \left\{ \overline{1, \, S_n^{\langle 2 \rangle} + 1} \right\} \qquad (20)$$

determine the matrix element

$$p_J^{\langle r \rangle} = K_r(\mathbf{X}, \mathbf{Y}), \; x_m = x_m^{\langle j_m \rangle}, \; y_n = y_n^{\langle j_{M+n} \rangle}, \; \forall \, m = \overline{1, M}, \; \forall \, n = \overline{1, N}. \qquad (21)$$

Then with the following assertion the finite TSNCG (12) is going to be mapped into BMG.

Theorem 2. Retrieved from the finite TSNCG (12), MDM $\mathbf{P}_r(0) = \left[p_J^{\langle r \rangle} \right]_F$ of the format (19) by (21) under indexing (20) is mapped into two-dimensional matrix $\mathbf{G}_r(0) = \left[g_{u_1 u_2}^{\langle r \rangle} \right]_{\mathscr{L}}$ of elements $g_{u_1 u_2}^{\langle r \rangle} = p_J^{\langle r \rangle}$ with ordinary two-positional indexing, whose format is

$$L = \left\{ \prod_{m=1}^{M} \left(S_m^{\langle 1 \rangle} + 1 \right) \right\} \times \left\{ \prod_{n=1}^{N} \left(S_n^{\langle 2 \rangle} + 1 \right) \right\}. \qquad (22)$$

The matrix map $\mathbf{P}_r(0) \to \mathbf{G}_r(0)$, realizing this reshaping, is reversible.

Proof. The first M indices in the set (20) of the matrix $\mathbf{P}_r(0) = \left[p_J^{\langle r \rangle} \right]_F$ element (21) correspond to components of the pure strategy of FP, and the last N ones correspond to components of the pure strategy of SP. Therefore, letting the subset of indices $\{ j_m \}_{m=1}^{M} \subset J$ be convolved into value

$$u_1 = \sum_{m=1}^{M} \left(\prod_{m_1=1}^{m-1} \left(S_{M-m_1+1}^{\langle 1 \rangle} + 1 \right) \right) \cdot \left(j_{M-m+1} - \operatorname{sign}(m-1) \right), \quad m = \overline{1, M} \qquad (23)$$

and letting the subset of indices $\{ j_{M+n} \}_{n=1}^{N} \subset J$ be convolved into value

$$u_2 = \sum_{n=1}^{N} \left(\prod_{n_1=1}^{n-1} \left(S_{N-n_1+1}^{\langle 2 \rangle} + 1 \right) \right) \cdot \left(j_{M+N-n+1} - \operatorname{sign}(n-1) \right), \quad n = \overline{1, N} \qquad (24)$$

the values (23) and (24) get integer. Moreover, for $j_m = \overline{1, S_m^{\langle 1 \rangle} + 1}$ there is

$$u_1 = \overline{1, Q_1(0)}, \; Q_1(0) = \prod_{m=1}^{M} \left(S_m^{\langle 1 \rangle} + 1 \right) \qquad (25)$$

and for $j_{M+n} = \overline{1, S_n^{\langle 2 \rangle} + 1}$ there is

$$u_2 = \overline{1, Q_2(0)}, \; Q_2(0) = \prod_{n=1}^{N} \left(S_n^{\langle 2 \rangle} + 1 \right). \qquad (26)$$

Thus $(M+N)$-dimensional matrix $\mathbf{P}_r(0) = \left[p_J^{\langle r \rangle} \right]_F$ of the format (19) by (20) and (21) is reshaped into two-dimensional matrix $\mathbf{G}_r(0) = \left[g_{u_1 u_2}^{\langle r \rangle} \right]_L$ of the format (22), where the set of indices $J = \{ j_k \}_{k=1}^{M+N}$ is mapped into the set $\{ u_1, u_2 \}$ by convolutions in (23) and (24). Further, let the function $\rho(a, b)$ by $b \neq 0$ round the fraction $\frac{a}{b}$ to the nearest integer towards zero. And let

$$\eta(a, b) = a - b \cdot \rho(a, b).$$

Then the subset of indices $\{ j_m \}_{m=1}^{M} \subset J$ is restored by the index u_1 after (23):

$$j_M = \eta\left(u_1, S_M^{\langle 1 \rangle} + 1 \right) + \left(S_M^{\langle 1 \rangle} + 1 \right) \left(1 - \operatorname{sign}\left[\eta\left(u_1, S_M^{\langle 1 \rangle} + 1 \right) \right] \right), \qquad (27)$$

$$j_{M-m} = 1 + \eta\left(\frac{u_1 - j_M - \sum_{m_1=1}^{m-1} \left(\prod_{m_2=1}^{m_1} \left(S_{M-m_2+1}^{\langle 1 \rangle} + 1 \right) \right) \cdot \left(j_{M-m_1} - 1 \right)}{\prod_{m_1=1}^{m} \left(S_{M-m_1+1}^{\langle 1 \rangle} + 1 \right)}, \; S_{M-m}^{\langle 1 \rangle} + 1 \right), \; \forall \, m = \overline{1, M-1}. \qquad (28)$$

The subset of indices $\{ j_{M+n} \}_{n=1}^{N} \subset J$ is restored by the index u_2 after (24) similarly:

$$j_{M+N} = \eta\left(u_2, S_N^{\langle 2 \rangle} + 1 \right) + \left(S_N^{\langle 2 \rangle} + 1 \right) \left(1 - \operatorname{sign}\left[\eta\left(u_2, S_N^{\langle 2 \rangle} + 1 \right) \right] \right), \qquad (29)$$

$$j_{M+N-n}=1+\eta\left(\frac{u_2-j_{M+N}-\sum_{n_1=1}^{n-1}\left(\prod_{n_2=1}^{n_1}\left(S_{N-n_2+1}^{\langle 2\rangle}+1\right)\right)\cdot\left(j_{M+N-n_1}-1\right)}{\prod_{n_1=1}^{n}\left(S_{N-n_1+1}^{\langle 2\rangle}+1\right)},\ S_{N-n}^{\langle 2\rangle}+1\right),\ \forall\ n=\overline{1,\ N-1}. \quad (30)$$

Consequently, the matrix map $\mathbf{P}_r(0)\to\mathbf{G}_r(0)$ is accomplished via (23) and (24), and the matrix map $\mathbf{G}_r(0)\to\mathbf{P}_r(0)$ is accomplished via (27) — (30). The theorem has been proved.

When $M=N=1$, the finite TSNCG (12) is BMG, and Theorem 2 is useless. When $M\neq 1$ or $N\neq 1$, Theorem 2 allows mapping the finite TSNCG (12) on FHCIL $D^{\langle 1\rangle}\times D^{\langle 2\rangle}$ into BMG.

$$\left\langle\left\{z_{u_1}^{\langle\mathbf{X}\rangle}(0)\right\}_{u_1=1}^{Q_1(0)},\ \left\{z_{u_2}^{\langle\mathbf{Y}\rangle}(0)\right\}_{u_2=1}^{Q_2(0)},\ \mathbf{G}_1(0),\ \mathbf{G}_2(0)\right\rangle \quad (31)$$

with the pure strategy of FP $z_{u_1}^{\langle\mathbf{X}\rangle}(0)$ corresponding to its strategy $\mathbf{X}=\left[x_m^{\langle j_m\rangle}\right]_{1\times M}\in D^{\langle 1\rangle}$ in the initial TSNCG (1) after having sampled under numbers $\left\{S_m^{\langle 1\rangle}\right\}_{m=1}^{M}$, and the pure strategy of SP $z_{u_2}^{\langle\mathbf{Y}\rangle}(0)$ corresponding to its strategy $\mathbf{Y}=\left[y_n^{\langle j_{M+n}\rangle}\right]_{1\times N}\in D^{\langle 2\rangle}$ in the initial TSNCG (1) after having sampled under numbers $\left\{S_n^{\langle 2\rangle}\right\}_{n=1}^{N}$. But before calling BMG (31) the approximation of TSNCG (1), accurateness of the approximation must be ranked.

6 Consistency of the players' equilibrium FSS in BMG

Denote by

$$\left\{\left\{p_*(u_1,\ 0)\right\}_{u_1=1}^{Q_1(0)},\ \left\{q_*(u_2,\ 0)\right\}_{u_2=1}^{Q_2(0)}\right\} \quad (32)$$

a Nash equilibrium solution or Pareto efficiency solution or other type equilibrium solution in BMG (31), in which $p_*(u_1,\ 0)$ is the optimal probability of applying the pure strategy $z_{u_1}^{\langle\mathbf{X}\rangle}(0)$, and $q_*(u_2,\ 0)$ is the optimal probability of applying the pure strategy $z_{u_2}^{\langle\mathbf{Y}\rangle}(0)$. In ranking the approximation accurateness, the solution (32) should be compared to solutions of other BMG, approximating the initial TSNCG (1). Their formats differ from the format (22) because these BMG are built under the sampling numbers

$$\left\{\left\{S_m^{\langle 1\rangle}+\delta\right\}_{m=1}^{M},\ \left\{S_n^{\langle 2\rangle}+\delta\right\}_{n=1}^{N}\right\},\ \delta\in\mathbb{Z}\setminus\{0\} \quad (33)$$

instead of (14). This is the only way to get comparisons because the genuine solution of the initial TSNCG (1) is often cannot be known. A new BMG is

$$\left\langle\left\{z_{u_1}^{\langle\mathbf{X}\rangle}(\delta)\right\}_{u_1=1}^{Q_1(\delta)},\ \left\{z_{u_2}^{\langle\mathbf{Y}\rangle}(\delta)\right\}_{u_2=1}^{Q_2(\delta)},\ \mathbf{G}_1(\delta),\ \mathbf{G}_2(\delta)\right\rangle \quad (34)$$

by

$$Q_1(\delta)=\prod_{m=1}^{M}\left(S_m^{\langle 1\rangle}+1+\delta\right),\ Q_2(\delta)=\prod_{n=1}^{N}\left(S_n^{\langle 2\rangle}+1+\delta\right),$$

and δ-BMG (34) is built under the sampling numbers (33) and re-finding (8), (9), and re-mapping $\mathbf{P}_r(\delta)\to\mathbf{G}_r(\delta)$ with identifications

$$\left\{S_m^{\langle 1\rangle}\equiv S_m^{\langle 1\rangle}+\delta\right\}_{m=1}^{M},\ \left\{S_n^{\langle 2\rangle}\equiv S_n^{\langle 2\rangle}+\delta\right\}_{n=1}^{N}, \quad (35)$$

whereupon the pure strategy of FP $z_{u_1}^{\langle\mathbf{X}\rangle}(\delta)$ corresponds to its strategy $\mathbf{X}=\left[x_m^{\langle j_m\rangle}\right]_{1\times M}$ in the initial TSNCG (1) after having sampled under numbers $\left\{S_m^{\langle 1\rangle}+\delta\right\}_{m=1}^{M}$, and the pure strategy of SP $z_{u_2}^{\langle\mathbf{Y}\rangle}(\delta)$ corresponds to its strategy $\mathbf{Y}=\left[y_n^{\langle j_{M+n}\rangle}\right]_{1\times N}$ in the initial TSNCG (1) after having sampled under numbers $\left\{S_n^{\langle 2\rangle}+\delta\right\}_{n=1}^{N}$. And may there be a convention that the sampling numbers (33) are chosen against the numbers (14) so that density of the sampling points along each dimension by $\delta>0$ doesn't decrease, and density of the sampling points along each dimension by $\delta<0$ doesn't increase. That is, for points

$$\left\{\left\{\left\{x_m^{\langle s_m\rangle}(\delta)\right\}_{s_m=1}^{S_m^{\langle 1\rangle}+1}\right\}_{m=1}^{M},\ \left\{\left\{y_n^{\langle s_n\rangle}(\delta)\right\}_{s_n=1}^{S_n^{\langle 2\rangle}+1}\right\}_{n=1}^{N}\right\} \quad (36)$$

chosen after the sampling numbers (33) with $\delta\in\mathbb{N}$, the inequalities

$$\max_{d_m=\overline{1,\ S_m^{\langle 1\rangle}}}\left(x_m^{\langle d_m+1\rangle}-x_m^{\langle d_m\rangle}\right)\geqslant\max_{d_m=\overline{1,\ S_m^{\langle 1\rangle}+\delta}}\left(x_m^{\langle d_m+1\rangle}(\delta)-x_m^{\langle d_m\rangle}(\delta)\right)$$
$$m=\overline{1,\ M} \quad (37)$$

and

$$\max_{d_n=\overline{1,\ S_n^{\langle 2\rangle}}}\left(y_n^{\langle d_n+1\rangle}-y_n^{\langle d_n\rangle}\right)\geqslant\max_{d_n=\overline{1,\ S_n^{\langle 2\rangle}+\delta}}\left(y_n^{\langle d_n+1\rangle}(\delta)-y_n^{\langle d_n\rangle}(\delta)\right)$$
$$n=\overline{1,\ N} \quad (38)$$

hold.

Formally, BMG (31) is a particular case of BMG (34), taken by $\delta=0$. Denote the solution of BMG (34) by

$$\left\{\left\{p_*(u_1,\ \delta)\right\}_{u_1=1}^{Q_1(\delta)},\ \left\{q_*(u_2,\ \delta)\right\}_{u_2=1}^{Q_2(\delta)}\right\} \quad (39)$$

similarly to denotation (32), in which $p_*(u_1, \delta)$ is the optimal probability of applying the pure strategy $z_{u_1}^{\langle \mathbf{X} \rangle}(\delta)$ and $q_*(u_2, \delta)$ is the optimal probability of applying the pure strategy $z_{u_2}^{\langle \mathbf{Y} \rangle}(\delta)$. Thus by denoting supports

$$\operatorname{supp}\left\{p_*(u_1, \delta)\right\}_{u_1=1}^{Q_1(\delta)} = \left\{z_{u_1^*}^{\langle \mathbf{X} \rangle}(\delta)\right\}_{u_1^* \in U_1(\delta) \subset \left\{\overline{1, Q_1(\delta)}\right\}} \tag{40}$$

and

$$\operatorname{supp}\left\{q_*(u_2, \delta)\right\}_{u_2=1}^{Q_2(\delta)} = \left\{z_{u_2^*}^{\langle \mathbf{Y} \rangle}(\delta)\right\}_{u_2^* \in U_2(\delta) \subset \left\{\overline{1, Q_2(\delta)}\right\}} \tag{41}$$

the r-th player gets payoff

$$\begin{aligned} v_r^*(\delta) &= \sum_{u_1=1}^{Q_1(\delta)} \sum_{u_2=1}^{Q_2(\delta)} g_{u_1 u_2}^{\langle r \rangle} \cdot p_*(u_1, \delta) \cdot q_*(u_2, \delta) = \\ &= \sum_{u_1^* \in U_1(\delta)} \sum_{u_2^* \in U_2(\delta)} g_{u_1^* u_2^*}^{\langle r \rangle} \cdot p_*(u_1^*, \delta) \cdot q_*(u_2^*, \delta) \end{aligned} \tag{42}$$

in situation (39), $r \in \{1, 2\}$. Henceforward, BMG (31) can be compared to BMG (34) by $\delta \in \mathbb{Z} \setminus \{0\}$ in two ways: through comparing payoffs $\left\{v_r^*(0)\right\}_{r \in \{1, 2\}}$ and $\left\{\left\{v_r^*(\delta)\right\}_{r \in \{1, 2\}}\right\}_{\delta \in \mathbb{Z} \setminus \{0\}}$, and through comparisons among supports (40) and among supports (41). The second comparison way relates to the support cardinality comparisons, and to the support configuration comparisons regarding the hypercube of the player's pure strategies.

The narrowest comparison takes $\delta \in \{-1, 1\}$. Namely, within minimal neighborhood of the sampling numbers (14), the solution of BMG mustn't vary much. And it is clear that the solution (32) can be put for consideration as the approximate solution of the initial TSNCG (1) if

$$|U_r(1)| \geq |U_r(0)|, \; r \in \{1, 2\} \tag{43}$$

and

$$\left|v_r^*(0) - v_r^*(1)\right| \leq \left|v_r^*(-1) - v_r^*(0)\right|, \; r \in \{1, 2\}. \tag{44}$$

The inequalities (43) and (44) reflect both the payoff comparison side and the support cardinality comparison side. And the last is strengthened involving (-1)-BMG:

$$|U_r(1)| \geq |U_r(0)| \geq |U_r(-1)|, \; r \in \{1, 2\}. \tag{45}$$

However, there are no comparisons among the support configurations. These comparisons are really needed because configuration of FSS may differ significantly from the genuine equilibrium strategy support in TSNCG (1) genuine solution. Also it may differ from the configuration of the support, obtained after different sampling. Hence, configuration of the player's FSS mustn't vary much as the sampling numbers change consentaneously.

For seeing the configuration of the players' equilibrium FSS in BMG (31) solution (32), they are going to be represented as piecewise linear hypersurfaces $\left\{h_r(u_r, 0)\right\}_{r \in \{1, 2\}}$, whose nonzero vertices are those equilibrium FSS probabilities linearly linked to points on FHCIL not included into the support. The hypersurfaces in δ-BMG (34) are denoted $\left\{h_r(u_r, \delta)\right\}_{r \in \{1, 2\}}$. Vertices of FP hypersurface $h_1(u_1, 0)$ are in points

$$\left\{\left\{x_m^{\langle j_m \rangle}\right\}_{m=1}^{M}, \; p_*(u_1, 0)\right\}_{u_1=1}^{Q_1(0)} \tag{46}$$

in the space $\mathbb{R}^{M+1}$, and vertices of SP hypersurface $h_2(u_2, 0)$ are in points

$$\left\{\left\{y_n^{\langle j_{M+n} \rangle}\right\}_{n=1}^{N}, \; q_*(u_2, 0)\right\}_{u_2=1}^{Q_2(0)} \tag{47}$$

in the space $\mathbb{R}^{N+1}$.

FP matches the index $u_1^* \in U_1(0)$ to the point

$$\mathbf{X}_q(0) = \left[x_m^{\langle q \rangle}(0)\right]_{1 \times M} = \left[x_m^{\langle j_m(q, 0) \rangle}\right]_{1 \times M} \in H_1, \quad q = \overline{1, Q_1^*(0)} \tag{48}$$

at $Q_1^*(0) = |U_1(0)|$ through expanding the index u_1^* via (27) and (28) into subset $\left\{j_m(q, 0)\right\}_{m=1}^{M} \subset J$. SP matches the index $u_2^* \in U_2(0)$ to the point

$$\mathbf{Y}_w(0) = \left[y_n^{\langle w \rangle}(0)\right]_{1 \times N} = \left[y_n^{\langle j_{M+n}(w, 0) \rangle}\right]_{1 \times N} \in H_2, \quad w = \overline{1, Q_2^*(0)} \tag{49}$$

at $Q_2^*(0) = |U_2(0)|$ through expanding the index u_2^* via (29) and (30) into subset $\left\{j_{M+n}(w, 0)\right\}_{n=1}^{N} \subset J$. Then let the set $\left\{\mathbf{X}_q(0)\right\}_{q=1}^{Q_1^*(0)}$ of the points (48) be sorted into the set

$$\begin{aligned} \left\{\bar{\mathbf{X}}_q(0)\right\}_{q=1}^{Q_1^*(0)} &= \left\{\left[x_m^{\langle \bar{j}_m(q, 0) \rangle}\right]_{1 \times M}\right\}_{q=1}^{Q_1^*(0)} \cap \left\{\mathbf{X}_q(0)\right\}_{q=1}^{Q_1^*(0)} = \\ &= \left\{\mathbf{X}_q(0)\right\}_{q=1}^{Q_1^*(0)} \subset H_1 \end{aligned} \tag{50}$$

so that the value

$$\begin{aligned} &\min_{q_1 \in \left\{\overline{q+1, Q_1^*(0)}\right\}} \rho_{\mathbb{R}^M}\left(\bar{\mathbf{X}}_q(0) - \bar{\mathbf{X}}_{q_1}(0)\right) = \\ &= \min_{q_1 \in \left\{\overline{q+1, Q_1^*(0)}\right\}} \left\|\bar{\mathbf{X}}_q(0) - \bar{\mathbf{X}}_{q_1}(0)\right\| = \\ &= \min_{q_1 \in \left\{\overline{q+1, Q_1^*(0)}\right\}} \sqrt{\sum_{m=1}^{M}\left(x_m^{\langle \bar{j}_m(q, 0) \rangle} - x_m^{\langle \bar{j}_m(q_1, 0) \rangle}\right)^2} \end{aligned} \tag{51}$$

with the re-sorted subset $\left\{\bar{j}_m(q, 0)\right\}_{m=1}^{M} \cap \left\{j_m(q, 0)\right\}_{m=1}^{M} = \left\{j_m(q, 0)\right\}_{m=1}^{M} \subset J$ is reached at $q_1 = q+1$ for each $q = \overline{1, Q_1^*(0) - 1}$ by $Q_1^*(0) < Q_1(0)$. Similarly, the set $\left\{\mathbf{Y}_w(0)\right\}_{w=1}^{Q_2^*(0)}$ of the points (49) is sorted into the set

$$\left\{\bar{\mathbf{Y}}_w(0)\right\}_{w=1}^{Q_2^*(0)} = \left\{\left[y_n^{\langle \bar{j}_{M+n}(w,\,0)\rangle}\right]_{1\times N}\right\}_{w=1}^{Q_2^*(0)} \cap \left\{\mathbf{Y}_w(0)\right\}_{w=1}^{Q_2^*(0)} = \\ = \left\{\mathbf{Y}_w(0)\right\}_{w=1}^{Q_2^*(0)} \subset H_2 , \quad (52)$$

so that the value

$$\min_{w_1\in\left\{\overline{w+1,\, Q_2^*(0)}\right\}} \rho_{\mathbb{R}^N}\left(\bar{\mathbf{Y}}_w(0) - \bar{\mathbf{Y}}_{w_1}(0)\right) = \\ = \min_{w_1\in\left\{\overline{w+1,\, Q_2^*(0)}\right\}} \left\|\bar{\mathbf{Y}}_w(0) - \bar{\mathbf{Y}}_{w_1}(0)\right\| = \\ = \min_{w_1\in\left\{\overline{w+1,\, Q_2^*(0)}\right\}} \sqrt{\sum_{n=1}^{N}\left(y_n^{\langle \bar{j}_{M+n}(w,\,0)\rangle} - y_n^{\langle \bar{j}_{M+n}(w_1,\,0)\rangle}\right)^2} , \quad (53)$$

with the re-sorted subset

$$\left\{\bar{j}_{M+n}(w,\,0)\right\}_{n=1}^{N} \cap \left\{j_{M+n}(w,\,0)\right\}_{n=1}^{N} = \\ = \left\{j_{M+n}(w,\,0)\right\}_{n=1}^{N} \subset J \quad (54)$$

is reached at $w_1 = w+1$ for each $w = \overline{1,\, Q_2^*(0)-1}$ by $Q_2^*(0) < Q_2(0)$. Importantly, one ought to be aware of that the result of sorting in (50) and (52) depends on selection of the initial points $\bar{\mathbf{X}}_1(0) \in \left\{\mathbf{X}_q(0)\right\}_{q=1}^{Q_1^*(0)}$ and $\bar{\mathbf{Y}}_1(0) \in \left\{\mathbf{Y}_w(0)\right\}_{w=1}^{Q_2^*(0)}$. In the case of completely mixed strategies, let them be

$$\bar{\mathbf{X}}_q(0) = \mathbf{X}_q(0),\ \forall\ q = \overline{1,\, Q_1^*(0)},\ Q_1^*(0) = Q_1(0) \quad (55)$$

and

$$\bar{\mathbf{Y}}_w(0) = \mathbf{Y}_w(0),\ \forall\ w = \overline{1,\, Q_2^*(0)},\ Q_2^*(0) = Q_2(0) \quad (56)$$

for the sake of convenience.

Considering δ -BMG (34), let the hypersurfaces $\left\{h_r(u_r,\,\delta)\right\}_{r\in\{1,\,2\}}$ and sets $\left\langle \left\{\bar{\mathbf{X}}_q(\delta)\right\}_{q=1}^{Q_1^*(\delta)},\ \left\{\bar{\mathbf{Y}}_w(\delta)\right\}_{w=1}^{Q_2^*(\delta)} \right\rangle$ regard built and found with identifications (35) and turning to these sets' description for (46) — (56). Thus, the second comparison way opens for the support configuration comparisons. And then there is a way to learn the rank of accurateness in approximating the initial TSNCG (1) with BMG (31).

Definition 1. The solution (32) of BMG (31) is called weakly consistent for being the approximate solution of TSNCG (1) if the inequalities

$$\max_{q\in\left\{\overline{1,\, Q_1^*(1)-1}\right\}} \rho_{\mathbb{R}^M}\left(\bar{\mathbf{X}}_q(1), \bar{\mathbf{X}}_{q+1}(1)\right) \leq \\ \leq \max_{q\in\left\{\overline{1,\, Q_1^*(0)-1}\right\}} \rho_{\mathbb{R}^M}\left(\bar{\mathbf{X}}_q(0), \bar{\mathbf{X}}_{q+1}(0)\right) , \quad (57)$$

$$\max_{w\in\left\{\overline{1,\, Q_2^*(1)-1}\right\}} \rho_{\mathbb{R}^N}\left(\bar{\mathbf{Y}}_w(1), \bar{\mathbf{Y}}_{w+1}(1)\right) \leq \\ \leq \max_{w\in\left\{\overline{1,\, Q_2^*(0)-1}\right\}} \rho_{\mathbb{R}^N}\left(\bar{\mathbf{Y}}_w(0), \bar{\mathbf{Y}}_{w+1}(0)\right) , \quad (58)$$

$$\max_{H_r}\left|h_r(u_r, 0) - h_r(u_r, 1)\right| \leq \\ \leq \max_{H_r}\left|h_r(u_r, -1) - h_r(u_r, 0)\right| , \quad (59)$$

and

$$\left\|h_r(u_r, 0) - h_r(u_r, 1)\right\| \leq \left\|h_r(u_r, -1) - h_r(u_r, 0)\right\|, \\ L_2(H_r),\ r\in\{1, 2\} \quad 60)$$

are true along with (43) and (44). Then solution (32) of BMG (31) is called weakly 1-consistent. Every strategy and its support in the weakly 1-consistent solution are called weakly consistent or weakly 1-consistent.

Weak 1-consistency of the players' equilibrium FSS in BMG (31) invokes minimal number of δ -BMG, approximating the initial TSNCG (1). This is the most primitive consistency in ranking the approximation accurateness. The primitiveness of Definition 1 is obviated with adding conditions of the sampling minimal loosing.

Definition 2. The weakly consistent solution (32) of BMG (31) is called consistent for being the approximate solution of TSNCG (1) if the inequalities (45) and

$$\max_{q\in\left\{\overline{1,\, Q_1^*(0)-1}\right\}} \rho_{\mathbb{R}^M}\left(\bar{\mathbf{X}}_q(0), \bar{\mathbf{X}}_{q+1}(0)\right) \leq \\ \leq \max_{q\in\left\{\overline{1,\, Q_1^*(-1)-1}\right\}} \rho_{\mathbb{R}^M}\left(\bar{\mathbf{X}}_q(-1), \bar{\mathbf{X}}_{q+1}(-1)\right) , \quad (61)$$

and

$$\max_{w\in\left\{\overline{1,\, Q_2^*(0)-1}\right\}} \rho_{\mathbb{R}^N}\left(\bar{\mathbf{Y}}_w(0), \bar{\mathbf{Y}}_{w+1}(0)\right) \leq \\ \leq \max_{w\in\left\{\overline{1,\, Q_2^*(0)-1}\right\}} \rho_{\mathbb{R}^N}\left(\bar{\mathbf{Y}}_w(-1), \bar{\mathbf{Y}}_{w+1}(-1)\right) \quad (62)$$

are true. Then solution (32) of BMG (31) is called 1-consistent. Every strategy and its support in the 1-consistent solution are called consistent or 1-consistent.

Inequalities (45), (61), (62), canceling the "weakness" in consistency, mean that the properties of the solution of (-1) -BMG relate to the properties of the solution (32) of BMG (31) as similarly as the properties of the solution (32) of BMG (31) relate to the properties of the solution of 1 -BMG. Note that in controlling the players' equilibrium FSS for their weak 1-consistency, there are 10 inequalities (43), (44), and (57) — (60) to be checked. And there are 14 inequalities (44), (45), and (57) — (62) to be checked for controlling the players' equilibrium FSS for their 1-consistency. Below is an opportunity to avoid superfluous computations in checking weak 1-consistency.

Theorem 3. If the solution

$$\left\langle \left\{p_*(u_1,\,1)\right\}_{u_1=1}^{Q_1(1)},\ \left\{q_*(u_2,\,1)\right\}_{u_2=1}^{Q_2(1)} \right\rangle \quad (63)$$

of 1 -BMG is completely mixed, then for checking weak 1-

consistency of the solution (32) it is sufficient to check six inequalities (44) and (59), (60).

Proof. Inasmuch as the situation (63) is completely mixed then

$$Q_1^*(1)=Q_1(1)=\prod_{m=1}^{M}\left(S_m^{\langle 1\rangle}+2\right)>\prod_{m=1}^{M}\left(S_m^{\langle 1\rangle}+1\right)\geq Q_1^*(0),$$

$$Q_2^*(1)=Q_2(1)=\prod_{n=1}^{N}\left(S_n^{\langle 2\rangle}+2\right)>\prod_{n=1}^{N}\left(S_n^{\langle 2\rangle}+1\right)\geq Q_2^*(0),$$

giving us both the inequalities (43), even strictly. Further, as the solution (63) is completely mixed then through the convention (55) and (56) there are the sets

$$\left\{\overline{\mathbf{X}}_q(1)\right\}_{q=1}^{Q_1^*(1)}=\left\{\mathbf{X}_q(1)\right\}_{q=1}^{Q_1(1)}$$

and

$$\left\{\overline{\mathbf{Y}}_w(1)\right\}_{w=1}^{Q_2^*(1)}=\left\{\mathbf{Y}_w(1)\right\}_{w=1}^{Q_2(1)}$$

such that

$$\begin{aligned}&\max_{q\in\left\{\overline{1,\,Q_1^*(1)-1}\right\}}\rho_{\mathbb{R}^M}\left(\overline{\mathbf{X}}_q(1),\overline{\mathbf{X}}_{q+1}(1)\right)=\\&=\max_{m=\overline{1,\,M}}\ \max_{d_m=\overline{1,\,S_m^{\langle 1\rangle}+1}}\left(x_m^{\langle d_m+1\rangle}(1)-x_m^{\langle d_m\rangle}(1)\right)\end{aligned}\tag{64}$$

and

$$\begin{aligned}&\max_{w\in\left\{\overline{1,\,Q_2^*(1)-1}\right\}}\rho_{\mathbb{R}^N}\left(\overline{\mathbf{Y}}_w(1),\overline{\mathbf{Y}}_{w+1}(1)\right)=\\&=\max_{n=\overline{1,\,N}}\ \max_{d_n=\overline{1,\,S_n^{\langle 2\rangle}+1}}\left(y_n^{\langle d_n+1\rangle}(1)-y_n^{\langle d_n\rangle}(1)\right)\end{aligned}\tag{65}$$

Then, due to (37) and (38), have

$$\begin{aligned}&\max_{q\in\left\{\overline{1,\,Q_1^*(1)-1}\right\}}\rho_{\mathbb{R}^M}\left(\overline{\mathbf{X}}_q(1),\overline{\mathbf{X}}_{q+1}(1)\right)\leq\\&\leq\max_{m=\overline{1,\,M}}\ \max_{d_m=\overline{1,\,S_m^{\langle 1\rangle}}}\left(x_m^{\langle d_m+1\rangle}-x_m^{\langle d_m\rangle}\right)\leq\\&\leq\max_{q\in\left\{\overline{1,\,Q_1^*(0)-1}\right\}}\sqrt{\sum_{m=1}^{M}\left(x_m^{\langle\bar{j}_m(q+1,\,0)\rangle}-x_m^{\langle\bar{j}_m(q,\,0)\rangle}\right)^2}=\\&=\max_{q\in\left\{\overline{1,\,Q_1^*(0)-1}\right\}}\rho_{\mathbb{R}^M}\left(\overline{\mathbf{X}}_q(0),\overline{\mathbf{X}}_{q+1}(0)\right)\end{aligned},$$

$$\begin{aligned}&\max_{w\in\left\{\overline{1,\,Q_2^*(1)-1}\right\}}\rho_{\mathbb{R}^N}\left(\overline{\mathbf{Y}}_w(1),\overline{\mathbf{Y}}_{w+1}(1)\right)\leq\\&\leq\max_{n=\overline{1,\,N}}\ \max_{d_n=\overline{1,\,S_n^{\langle 2\rangle}}}\left(y_n^{\langle d_n+1\rangle}-y_n^{\langle d_n\rangle}\right)\leq\\&\leq\max_{w\in\left\{\overline{1,\,Q_2^*(0)-1}\right\}}\sqrt{\sum_{n=1}^{N}\left(y_n^{\langle\bar{j}_{M+n}(w+1,\,0)\rangle}-y_n^{\langle\bar{j}_{M+n}(w,\,0)\rangle}\right)^2}=\\&=\max_{w\in\left\{\overline{1,\,Q_2^*(0)-1}\right\}}\rho_{\mathbb{R}^N}\left(\overline{\mathbf{Y}}_w(0),\overline{\mathbf{Y}}_{w+1}(0)\right)\end{aligned},$$

giving us both the inequalities (57) and (58). The theorem has been proved.

Consistency by either Definition 1 or Definition 2 of the player's equilibrium FSS in BMG (31), approximating the player's genuine equilibrium strategy in TSNCG (1), ranks accurateness of the approximation for elementary case. Naturally, the rank conception in the form of (weak) 1-consistency is easily widened to the form of (weak) λ-consistency by $\lambda\in\mathbb{N}$.

7 Approximation of TSNCG (1) in λ-consistency

Definition 3. The solution (32) of BMG (31) is called weakly λ-consistent for being the approximate solution of TSNCG (1) if the inequalities

$$\left|v_r^*(\mu)-v_r^*(\mu+1)\right|\leq\left|v_r^*(\mu+1)-v_r^*(\mu)\right|,\ r\in\{1,\,2\},\tag{66}$$

$$\left|U_r(\mu+1)\right|\geq\left|U_r(\mu)\right|,\ r\in\{1,\,2\},\tag{67}$$

$$\begin{aligned}&\max_{q\in\left\{\overline{1,\,Q_1^*(\mu+1)-1}\right\}}\rho_{\mathbb{R}^M}\left(\overline{\mathbf{X}}_q(\mu+1),\overline{\mathbf{X}}_{q+1}(\mu+1)\right)\leq\\&\leq\max_{q\in\left\{\overline{1,\,Q_1^*(\mu)-1}\right\}}\rho_{\mathbb{R}^M}\left(\overline{\mathbf{X}}_q(\mu),\overline{\mathbf{X}}_{q+1}(\mu)\right)\end{aligned},\tag{68}$$

$$\begin{aligned}&\max_{w\in\left\{\overline{1,\,Q_2^*(\mu+1)-1}\right\}}\rho_{\mathbb{R}^N}\left(\overline{\mathbf{Y}}_w(\mu+1),\overline{\mathbf{Y}}_{w+1}(\mu+1)\right)\leq\\&\leq\max_{w\in\left\{\overline{1,\,Q_2^*(\mu)-1}\right\}}\rho_{\mathbb{R}^N}\left(\overline{\mathbf{Y}}_w(\mu),\overline{\mathbf{Y}}_{w+1}(\mu)\right)\end{aligned},\tag{69}$$

$$\begin{aligned}&\max_{H_r}\left|h_r(u_r,\mu)-h_r(u_r,\mu+1)\right|\leq\\&\leq\max_{H_r}\left|h_r(u_r,\mu-1)-h_r(u_r,\mu)\right|\end{aligned},\tag{70}$$

and

$$\begin{gathered}\left\|h_r(u_r,\mu)-h_r(u_r,\mu+1)\right\|\leq\left\|h_r(u_r,\mu-1)-h_r(u_r,\mu)\right\|,\\ L_2(H_r),r\in\{1,2\}\end{gathered}\tag{71}$$

are true $\forall\ \mu=\overline{1-\lambda,\ \lambda-1}$ by $\lambda\in\mathbb{N}$. Every strategy and its support in the weakly λ-consistent solution are called weakly λ-consistent.

Definition 4. The weakly λ-consistent solution (32) of BMG (31) is called λ-consistent for being the approximate solution of TSNCG (1) if the inequalities

$$\left|U_r(\mu)\right|\geq\left|U_r(\mu-1)\right|,\ r\in\{1,\,2\}\tag{72}$$

and

$$\begin{aligned}&\max_{q\in\left\{\overline{1,\,Q_1^*(\mu)-1}\right\}}\rho_{\mathbb{R}^M}\left(\overline{\mathbf{X}}_q(\mu),\overline{\mathbf{X}}_{q+1}(\mu)\right)\leq\\&\leq\max_{q\in\left\{\overline{1,\,Q_1^*(\mu-1)-1}\right\}}\rho_{\mathbb{R}^M}\left(\overline{\mathbf{X}}_q(\mu-1),\overline{\mathbf{X}}_{q+1}(\mu-1)\right)\end{aligned}\tag{73}$$

and

$$\begin{aligned}&\max_{w\in\left\{\overline{1,\,Q_2^*(\mu)-1}\right\}}\rho_{\mathbb{R}^N}\left(\overline{\mathbf{Y}}_w(\mu),\overline{\mathbf{Y}}_{w+1}(\mu)\right)\leq\\&\leq\max_{w\in\left\{\overline{1,\,Q_2^*(\mu-1)-1}\right\}}\rho_{\mathbb{R}^N}\left(\overline{\mathbf{Y}}_w(\mu-1),\overline{\mathbf{Y}}_{w+1}(\mu-1)\right)\end{aligned}\tag{74}$$

are true $\forall\ \mu = \overline{1-\lambda,\ \lambda-1}$ by $\lambda \in \mathbb{N}$. Every strategy and its support in the λ-consistent solution are called λ-consistent.

For checking λ-consistency of the weakly λ-consistent solution, it is not of necessity to check all $8\lambda-4$ inequalities (72) — (74). It is sufficient to check four inequalities ever.

Theorem 4. If the inequalities

$$\left|U_r(1-\lambda)\right| \geq \left|U_r(-\lambda)\right|,\ r \in \{1,\ 2\} \tag{75}$$

and

$$\max_{q \in \left\{\overline{1,\ Q_1^*(1-\lambda)-1}\right\}} \rho_{\mathbb{R}^M}\left(\overline{\mathbf{X}}_q(1-\lambda), \overline{\mathbf{X}}_{q+1}(1-\lambda)\right) \leq \\ \leq \max_{q \in \left\{\overline{1,\ Q_1^*(-\lambda)-1}\right\}} \rho_{\mathbb{R}^M}\left(\overline{\mathbf{X}}_q(-\lambda), \overline{\mathbf{X}}_{q+1}(-\lambda)\right) \tag{76}$$

and

$$\max_{w \in \left\{\overline{1,\ Q_2^*(1-\lambda)-1}\right\}} \rho_{\mathbb{R}^N}\left(\overline{\mathbf{Y}}_w(1-\lambda), \overline{\mathbf{Y}}_{w+1}(1-\lambda)\right) \leq \\ \leq \max_{w \in \left\{\overline{1,\ Q_2^*(-\lambda)-1}\right\}} \rho_{\mathbb{R}^N}\left(\overline{\mathbf{Y}}_w(-\lambda), \overline{\mathbf{Y}}_{w+1}(-\lambda)\right) \tag{77}$$

are true for some $\lambda \in \mathbb{N}$ then the weakly λ-consistent solution (32) of BMG (31) is λ-consistent.

Proof. Inasmuch as the inequalities (67) — (69) are true $\forall\ \mu = \overline{1-\lambda,\ \lambda-1}$ then, having added the four inequalities (75) — (77) to them, there are true the inequalities (72) — (74) $\forall\ \mu = \overline{1-\lambda,\ \lambda-1}$ by $\lambda \in \mathbb{N}$. The theorem has been proved.

Apparently, weak $(\lambda-1)$-consistency follows weak λ-consistency, and $(\lambda-1)$-consistency follows λ-consistency. Approximation of TSNCG (1) in λ-consistency under the generalizing Definitions 3 and 4 prescribes the monotonic-like properties for $2\lambda+1$ of BMG (34) by $\delta = \overline{-\lambda,\ \lambda}$. The greater λ the wider neighborhood of the sampling is, and the more suitable BMG (31) for being called the approximation of TSNCG (1). In ranking the approximation accurateness, λ-consistency invokes $2\lambda+1$ δ-BMG, approximating the initial TSNCG (1). The greater λ the higher rank of accurateness of the approximation is.

8 Conclusions and possibilities for further work

It is noteworthy to say that neither weakly λ-consistent solution (32) of BMG (31), nor λ-consistent solution (32) of BMG (31) guarantee the faultlessness of the initial TSNCG (1) approximation as BMG (31), whatever $\lambda \in \mathbb{N}$ is. Nonetheless weakly λ-consistent solution being a particular case of λ-consistency forces the equilibrium FSS cardinality nondecreasing as the sampling numbers increase minimally. This is stated with (67), and it is reinforced with (72) for λ-consistency. Secondly, as the sampling numbers increase minimally, both the players' equilibrium payoffs and the players' FSS differentiate no more than at the lesser sampling numbers. This is stated with (66), (70), (71). And thirdly, density of points on FHCIL constituting FSS is forced nondecreasing as the sampling numbers increase minimally, what is stated with (68) and (69) for weak λ-consistency and is reinforced with (73) and (74) for λ-consistency.

Of course, we could call an equilibrium FSS λ-consistent if it satisfied either the conditions (66) — (71) or (66) — (74), but is its consistency ever followed with the other player's equilibrium FSS? Surely, there is no proof of that. Also there is no proof of that limits

$$\lim_{\delta \to \infty} v_r^*(\delta),\ r \in \{1,\ 2\} \tag{78}$$

exist and they coincide with the players' genuine equilibrium payoffs in the initial TSNCG (1). Besides, many other equilibrium situations may be in BMG (31) or TSNCG (1), giving diverse payoffs for players [26, 27]. The deficiency of hypersurfaces $\left\{h_r(u_r,\ \delta)\right\}_{r \in \{1,\ 2\}}$ is that there is no proof of that limits

$$\lim_{\delta \to \infty} h_r(u_r,\ \delta),\ r \in \{1,\ 2\} \tag{79}$$

exist and they coincide with the players' genuine equilibrium strategies in the initial TSNCG (1). These demerits are nonetheless disregarded due to that the suggested dimension-dependent irregular samplings and MDM reshaping allow to solve TSNCG approximately [21, 22, 28] as BMG, controlling the approximation accurateness rank with λ-consistency. Clearly, checking consistency must be started from weak 1-consistency.

The proved items do have their merits. In order to sample PPF under conditions (15) and (16) from the hypercube (4) down to FHCIL $D^{\langle 1 \rangle} \times D^{\langle 2 \rangle}$, Theorem 1 determines the choice of the sampling numbers (14) and points (17). When $M \neq 1$ or $N \neq 1$, Theorem 2 allows mapping the finite TSNCG (12) on FHCIL $D^{\langle 1 \rangle} \times D^{\langle 2 \rangle}$ into BMG (31). Superfluous computations in checking weak 1-consistency are avoided with Theorem 3, allowing to check six inequalities instead of 10 inequalities. And Theorem 4 allows to complete checking λ-consistency of the weakly λ-consistent solution on four inequalities (75) — (77).

The suggested approximation of TSNCG (1) is fulfilled in three stages: PPF are sampled, the sampled PPF as MDM are mapped into ordinary flat matrices, and the solution of the corresponding BMG is checked out for its consistency. If the solution (32) is not even weakly 1-consistent, then the sampling numbers (14) should be increased. Partial increment (along some dimensions of the hypercube, but not all of them) is not excluded. If any increment is impossible then FHCIL (10) and (11) must be formed otherwise, accumulating the sets (8) and (9) with some new points (17). Generally, the suggested approximation tool is applicable to both unit-hypercubic infinite TSNCG and games which are isomorphic to unit-hypercubic infinite TSNCG [1, 29].

Further work will be focused on building an efficient sorter for solving the problems (50) — (53). The case with strict consistency, when every sign "greater than or equal" and every sign "less than or equal" appear "greater than" and "less than", ought to be thought over for possible convergences in (78) and (79). And there are questions waiting for their answers:

1. Shall one player use its equilibrium FSS satisfying conditions of (weakly) λ-consistency if the other player's equilibrium FSS isn't (weakly) λ-consistent? Or if the other player's equilibrium FSS has lower rank of consistency, say, when it is (weakly) $(\lambda-1)$-consistent?

2. Is it possible to determine (weak) λ-consistency of the solution (32) if one player's equilibrium FSS satisfies conditions of (weakly) λ-consistency?

3. Are necessarily two different equilibrium situations in BMG λ-consistent if one of them is λ-consistent already?

These questions are motives for continuation of research of approximating infinite TSNCG. For noncooperative games, the suggested approximation approach of consistency is going to fit anywise.

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