Stochastic resonance induced by over-damped fractional Langevin equation with α-stable noise

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Abstract

Stochastic resonance phenomenon induced in a system described by over-damped fractional Langevin equation with α-stable noise is investigated. When there is no external α-stable noise, the stochastic resonance is observed in case of the fractional order less than one certain threshold. By applying α-stable noise, the influences of the noise intensity and characteristic exponent of α-stable noise on the occurrence of stochastic resonance phenomenon are characterized. We find that the proper noise intensity enlarges the peak value of output power spectrum which is significant for stochastic resonance. Adjusting the noise intensity, the behaviour of signal-to-noise ratio is non-monotonic and with a maximum value. Under the same conditions, the lower value of characteristic exponent of α-stable noise leads to the smaller noise intensity to achieve stochastic resonance.

Keywords: stochastic resonance, over-damped fractional Langevin equation, α-stable noise

1 Introduction

The fractional calculus has a long history since it was first described by G.W. Leibniz [1], it allows the derivatives or integrals to be any non-integer order. Fractional calculus has been applied in many situations, such as viscoelasticity, confined geometries, biological tissues, thermoelasticity and control system etc. [2-6]. The classical derivatives and integrals form of fractional calculus are defined by Riemann-Liouville and Caputo.

Stochastic resonance (SR) has been widely investigated during past decades. It is a nonlinear phenomenon where a signal can be enhanced by adding noise. In the classical SR theory, models based on integer-order equation and double-well potential is defined to describe the resonance, which is characterized by the flow over the potential barrier [7]. In such a system, the occurrence of a single-well escape is a result of competition between damping and excitation. But this SR phenomenon is expected to be more complicated in system with memory effect, which can be introduced by hidden variables of non-viscous damping [8].

Recently, the phenomenon of stochastic resonance in the fractional order systems was investigated. The authors claimed that the stable steady states can be changed by fractional order damping and then lead to single- or double-well resonance behaviour [9]. SR was also investigated in the under-damped fractional Langevin equation, the signal-to-noise (SNR) and output signal's spectrum is found being non-monotonic, that indicates the SR phenomenon occur [10]. The authors also investigate that the SR phenomenon appears in the fractional order system and this characteristic can be used to detect the weak signal [11]. Peng Hao etc. studied the SR in the over-damped bistable system with chirp signal, the results show that there is certain relation with the chirp signal frequency and SR [12].

In most of the previous studies of SR in fractional-order system, the noise was assumed to be white Gaussian noise. It is commonly used to describe various phenomena due to the Central Limit Theorem. White Gaussian noise is just an ideal case for fluctuations, however, in practical applications, non-Gaussian statistics is better to explain the additive noise, such as noise in communications channel and embedded wireless laptop transceivers [13], were found to be impulsive, so they cannot be characterized well using the Gaussian noise. The α-stable noise can describe the impulsive characteristic of noise much better, it can maintain the generation mechanism of natural noise and limit distribution of propagation conditions, match the actual data well. Gaussian noise is a particular example of it [14]. Its distribution follows heavy-tail stable law statistics with infinite variance; it can not only simulate the stable situation of noise, but also the impulsive status.

The interest to discuss α-stable noise in SR has just been started. The authors of [15] employ numerical methods to find the solution of stochastic Langevin equation and space fractional kinetic Equation, they studied the properties of the probability density function (PDF) of a bistable system driven by heavy tailed white symmetric Lévy noise. It is founded that in contrast to the bistable system driven by Gaussian noise, in the Lévy case, the positions of maxima of the stationary PDF do not

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coincide with the positions of minima of the bistable potential. Tomasz Srokowski [16] discussed with generalized Langevin equation with double-well potential, the probability density distributions converge with time to a distribution similar to a Gaussian but tails have a power-law form. The SR phenomenon is emerged by means of spectral amplification.

In this paper, we consider the SR phenomenon induced by over-damped fractional Langevin equation with α -stable noise, in addition, by a period driving force. When the rate of the jumping between the potential wells due to the α -stable noise coincides with the frequency of the oscillatory force, the SR is observed. We discuss this phenomenon and demonstrate the influence to SR by various sets of the model parameters, signal-to-noise (SNR) and power spectrum amplification are taken as characteristic.

The paper is organized as follows. In section II, we introduce the models and methods of over-damped fractional Langevin equation and α -stable noise. The related potential function and density function of α -stable distribution are obtained in different cases. In section III, the variation of output signal and power spectrum with noise intensity is analysed, and we demonstrate how model parameters such as characteristic exponent modify its properties, in particular the SNR function. Section IV is some discussions and conclusions.

2 Models and methods

Consider an over-damped fractional Langevin equation [15, 16] driven by α -stable noise

$$ {}_0^C D^p x(t) + \frac{dV(x)}{dx} = F_1(t) + E\eta(t) \ , \qquad (1) $$

where ${}_0^C D^p x(t)$ is the p order fractional order derivative to $x(t)$ by using Caputo's definition, and $0 < p < 1$, the Caputo's definition is written as

$$ {}_a D_t^p f(t) = \frac{1}{\Gamma(n-p)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{p-n+1}} d\tau \ . \qquad (2) $$

We take a lower limit $a = 0$ for the above definitions. $F_1(t) = A\cos(2\pi\omega t)$ is an external signal with amplitude A and frequency ω, and $\eta(t)$ denotes the α -stable noise with characteristic exponent α ($\alpha \in (0,2]$), which obey to the α -stable distribution. E is the intensity of α -stable noise. When $\alpha = 2$, $\eta(t)$ becomes a Gaussian noise. The potential function *V(x)* in the Equation (1) is defined as

$$ V(x) = -ax^2/2 + bx^4/4 \ (a > 0, b > 0) \qquad (3) $$

V(x) is a symmetric double-well potential, as shown in Figure 1. There are two minima located at $\pm x_m$, they are separated by a potential barrier with height $\Delta V = a^2/(4b)$. Without the extern periodic forcing or the forcing is too weak, the particle cannot roll periodically from one potential well into the other on. From Equation (2) the Caputo's definition of fractional order differential, we can see the fractional order p relates with the memory characteristic, the bigger p means the memory characteristic much less, when $p \to 1$, it turns into the integer differential order, which indicates totally loss memory, while $p \to 0$, it differential equal to constant 1, that indicates the same memory characteristic to the speed of each time.

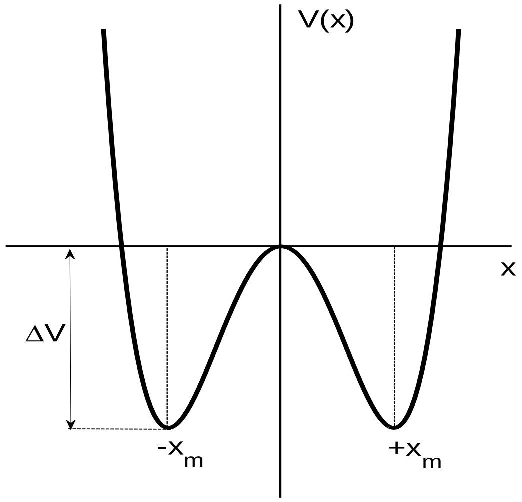
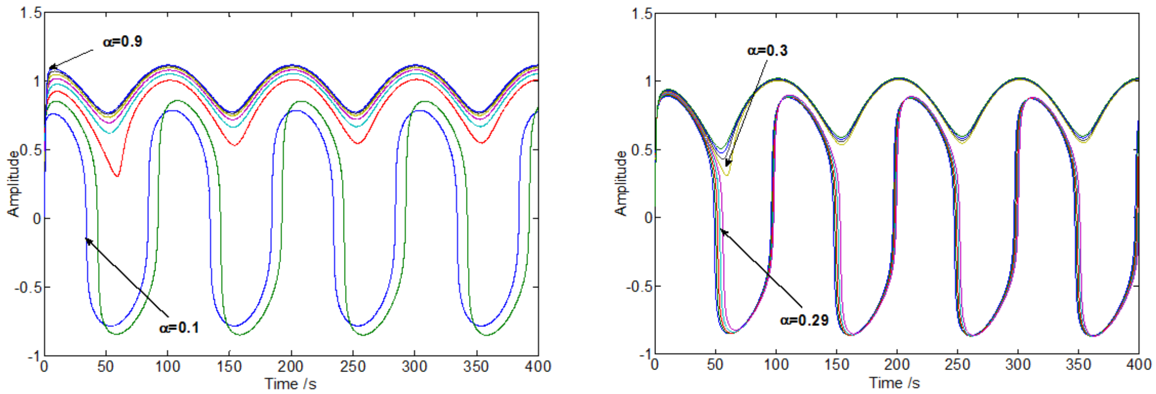


FIGURE 1 Sketch of the double-well potential V(x)

When there is no noise applied to the Equation (1), just in the presence of periodic driving force $A\cos(2\pi\omega t)$, by changing the fractional order p, the double-well potential is tilted back and forth, thereby the potential barriers of the right and the left well will be successively raised and lowered, respectively, in an anti-symmetric manner. Figure 2 shows the numerical simulation of Equation (1) without the noise $\eta(t)$, the other parameters are $a = b = 1, E = 0.3, f = 0.01$. Figure 2(a) indicates the curves when fractional order p changes from 0.9 to 0.1 with step 0.1; Figure 2(b) shows the curves when p changes from 0.3 to 0.2 with step 0.01. From the figures it can be concluded that as the fractional order p attenuated from (0,1). The particle is doing partial periodic motion around the balance point $x = 1$ or $x = -1$ with frequency $f = 0.01$, when p reaches a threshold p_t, the particles hop the potential barrier top which takes place at $x = 0$ into the other well, thus do the periodic motion between $x = \pm 1$ with the centre at $x = 0$. From the simulation results, the threshold p_t is 0.29. When p is less than p_t, the particles can hop the potential barrier top without external noise energy, thus the stochastic resonance phenomenon cannot be appeared. While p is greater than p_t, the particles just do partial periodic motion around one well, it only needs the synchronized action with the external noise to produce stochastic resonance phenomenon.



a) b)

FIGURE 2 Time domain of output signal with different fractional order *p*

α -stable noise $\eta(t)$ obeys α -stable distribution, whose characteristic function is [14]:

$$\varphi(t) = \begin{cases} \exp(-\gamma^{\alpha}|u|^{\alpha}[1 - i\beta(\tan\frac{\pi\alpha}{2})\text{sign}(u)] + i\delta\mu), \alpha \neq 1 \\ \exp(-\gamma|u|[1 + i\beta\frac{2}{\pi}\log|u|\,\text{sign}(u)] + i\delta\mu), \alpha = 1 \end{cases} ,(4)$$

where $\alpha \in (0,2]$, $\beta \in [-1,1]$, $\gamma \geq 0$ and $\delta \in \mathbb{R}$, $\text{sign}(u)$ function is -1 for a negative number, 0 for the number zero, or +1 for a position number. α -stable noise characteristic function is determined by four parameters: characteristic exponent α, scaling parameter γ, symmetry parameter β and location parameter δ. A small value of α will imply considerable probability mass in the tails of the distribution. It corresponds to the Gaussian distribution (for any β) when $\alpha = 2$, a Cauchy distribution with $\alpha = 1, \beta = 0$ and a Lévy distribution with $\alpha = 1/2, \beta = 1$.

Figure 3 illustrates the PDF of α -stable noise with different parameters. Figure 3(a) shows the relation of symmetric α -stable noise PDF with different characteristic exponent parameters α, Figure 3(b) displays the relation of skewed α -stable PDF with different symmetry parameters β.

In this paper the fractional order operator is approximated by a refined Oustaloup recursive filter [17] in a specified frequency range (ω_b, ω_h) and of order *N*. It is given by

$$s^{\alpha} \approx \left(\frac{d\omega_h}{b}\right)^{\alpha}\left(\frac{ds^2 + b\omega_h s}{d(1-\alpha)s^2 + b\omega_h s + d\alpha}\right)G_p, \qquad (5)$$

where G_p, ω_k, $\omega_k^{'}$ can be computed from

$$G_p = \prod_{k=-N}^{N}\frac{s+\omega_k^{'}}{s+\omega_k},\ \omega_k = (b\omega_h/d)^{\frac{\alpha+2k}{2N+1}},\ \omega_k^{'} = (b\omega_b/d)^{\frac{\alpha-2k}{2N+1}}.$$

A good approximation is obtained with *b*=10, *d*=9.

3 Numerical results

We fix the parameters in Equation (1) as $a = b = 1$, $A = 0.3$, $\omega = 0.01$, $\alpha = 1.5$, $\beta = 0.5$, $\gamma = 1.0$, $\delta = 0$ for α -stable noise $\eta(t)$, Equation (1) turns into:

$$\frac{d^p x(t)}{dt} = x - x^3 + 0.3\cos(2\pi \times 0.01) + E\eta(t) \qquad (6)$$

where *E* is the noise intensity.

3.1 THE VARIATION OF OUTPUT SIGNAL AND POWER SPECTRUM WITH NOISE INTENSITY

First we investigate the effects of α -stable noise on the evolution of *x*(*t*) under fractional order *p*=0.75. We choose the noise intensity *E*=1.5, The time domain and frequency spectrum of input signal and output signal *x*(*t*) are shown in Figure 3. Figure 4(a) shows the time domain of input signal which is the external periodic signal with α -stable noise, some sharp spikes are visible for the heavy tails of α -stable noise, Figure 4(c) illustrates the power spectrum of input signal, the peak amplitude of power spectrum is 0.1732 at frequency 0.01. Figure 4(b) shows the time domain of output signal, Figure 4(d) illustrates the power spectrum of output signal, the peak amplitude of power spectrum is 0.5198 at frequency 0.01, greater than 0.1732, which shows the SR significantly occurred. From Section II, we found that if there is no noise applied to the system, when fractional order *p* is greater than p_t, no SR phenomenon happened. From Figure 4 we know that the α -stable noise with proper intensity can cause the hopping of the particle between two potential wells, thus lead to the SR effect.

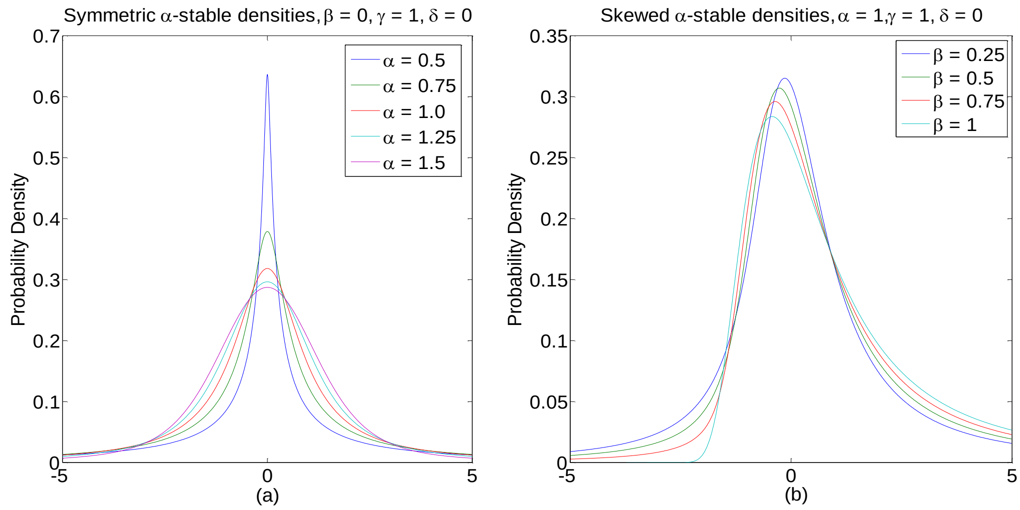


FIGURE 3 α-stable probability density functions

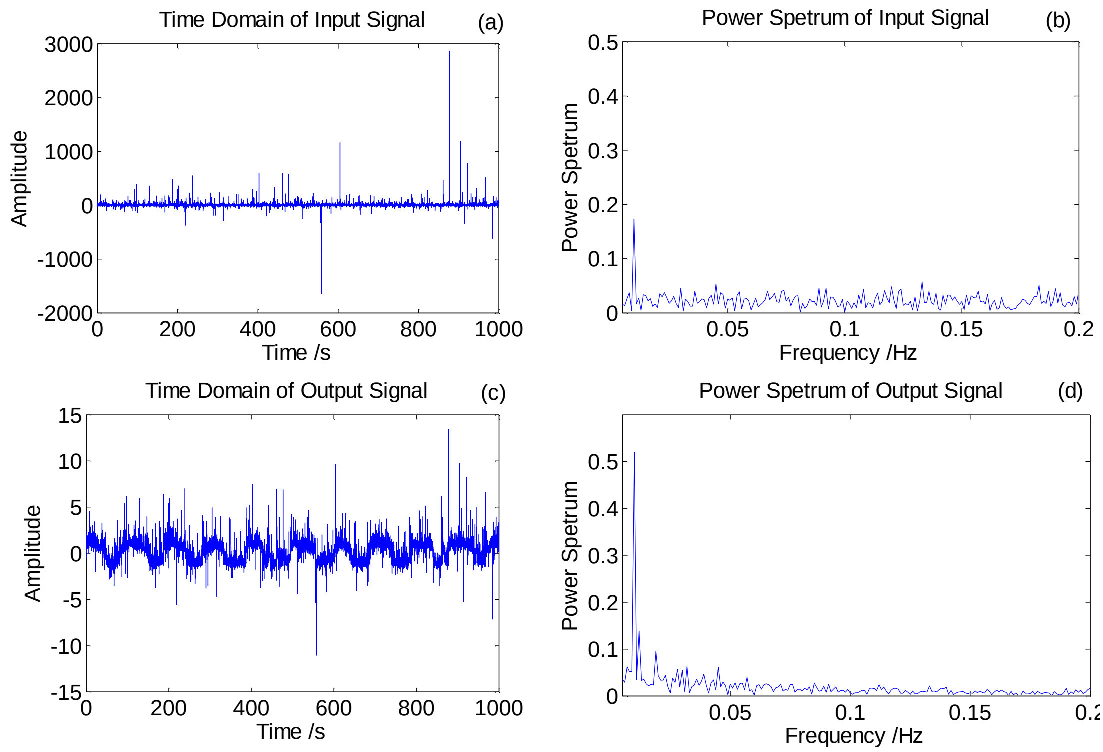


FIGURE 4 The time domain and power spectrum of input and output signal when intensity E=1.5

Figure 5 illustrates the statistics of particle oscillating back and forth between wells, it can be seen that the SR phenomenon is the enhancement of output signal via tuning the noise intensity, but when the noise intensity is bigger enough, the effect of SR gradually diminish. At a lower noise level, the particles oscillates at the minima of the potential wells for a long time and rarely switches between two potential wells, thus the periodic particles can hardly be visible at the other potential well. Under this circumstances, the periodic component of the output signal x(t) is primarily doing motion around the potential minima, which is interval motion in stochastic resonance, it is illustrated by Figure 5(a) where noise intensity E=0.7. However, when the noise intensity is increased to a certain value, the input-output synchronization effect happens, the periodic particles doing motion between two potential wells, we call it the interval motion in stochastic resonance. Figure 5(b) illustrated this phenomenon. Figure 5(c) shows that the synchronization vanishes when noise intensity is larger enough, that means the system flips too many times between its stable states within each forcing period, thus statistically irrelevant.

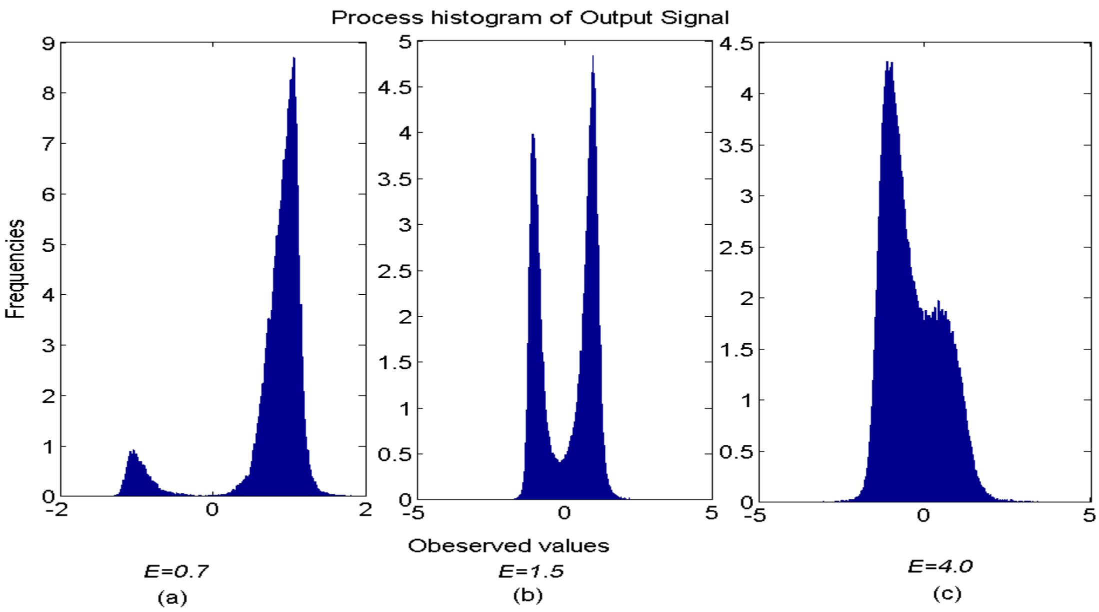


FIGURE 5 The process histogram of output signal

3.2 THE SNR FUNCTIONS WITH DIFFERENT CHARACTERISTIC EXPONENT

SNR is often taken as one quantitative indicator to demonstrate the SR phenomenon [18]. The definition is:

$$SNR = \frac{1}{S_N(\Omega)} \lim_{\Delta\omega\to\infty} \int_{\Omega-\Delta\omega}^{\Omega+\Delta\omega} S(\omega)d\omega \,, \tag{7}$$

where $\int_{\Omega-\Delta\omega}^{\Omega+\Delta\omega} S(\omega)d\omega$ represents the power carried by the signal, $S_N(\Omega)$ represents the noise power spectrum near the frequency, and $S(\omega)$ denotes the power spectrum of signal.

Figure 6 shows the SNR versus the noise intensity E with different characteristic exponent of α-stable noise, the other parameters being kept same. The values of SNR decrease with the noise intensity E at first, then begin to increase, and when the noise intensity reach to a critical value E_{SR}, the values of SNR achieve a maximum and after that decrease again. Under the different characteristic exponent α, the SNR is clearly non-monotonic, thus indicates the occurrence of SR phenomenon. As increasing the characteristic exponent α the SNR shifts towards bigger values of noise intensity.

4 Discussions and conclusions

In this paper, the properties of over-damped fractional Langevin equation with α-stable noise have been studied. In case of no external α-stable noise, the stochastic resonance phenomenon is observed when fractional order is less than one certain threshold. When fractional order is greater than the certain threshold, the SR is not appeared. However, by applying the α-stable noise, even at the situation with larger fractional order, the SR phenomenon is occurred, by comparing with the output power spectrum of input signal and output signal, we investigate that the proper noise intensity enhance the peak value of output power spectrum, the behaviour of SNR is non-monotonic, there is a maximum value when the noise intensity changes, thus is the typical SR phenomenon. We also find that at the same conditions, the smaller of the characteristic exponent of α-stable noise, the lower of noise intensity to achieve the SR.

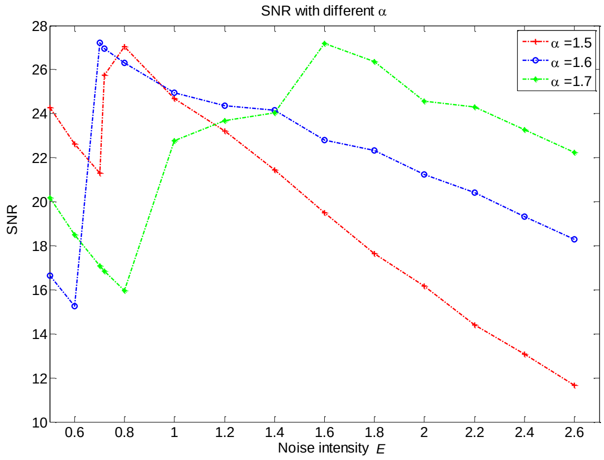


FIGURE 6 SNR versus noise intensity E with different α

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