Birkhoff normal forms for the wave equations with nonlinear terms depending on the time and space variables

Yi Wang*

School of Mathematics and Quantitative Economic, Shandong University of Finance and Economics, Jinan, Shandong 250014, P. R. China

Received 1 March 2014, www.tsi.lv

Abstract

The one-dimensional (1D) quasi-periodically forced nonlinear wave equation with periodic boundary conditions is considered. It is proved that there is a real analytic and symplectic change of coordinates, which can transform the Hamiltonian to the Birkhoff normal form.

Keywords: Infinite dimensional Hamiltonian system, quasi-periodically forced nonlinear wave equation, quasi-periodic solution, periodic boundary condition, Birkhoff normal form

1 Introduction and main results

In this paper, we are concerned with the quasi-periodically forced nonlinear wave equation

$$u_{tt} - u_{xx} + \mu u + \varepsilon g(\omega t, x) h(u) = 0, \quad \mu > 0, \quad x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \tag{1.1}$$

under the periodic boundary conditions

$$u(t,x) = u(t, x+2\pi), \tag{1.2}$$

where ε is a small positive parameter; the function $g(\omega t, x) = g(\vartheta, x)$, $(\vartheta, x) \in \mathbb{T}^m \times \mathbb{T}$ is real analytic in (ϑ, x) and quasi-periodic in t with frequency vectors $\omega = (\omega_1, \omega_2 \dots, \omega_m) \in [\varrho, 2\varrho]^m$ for some constant $\varrho > 0$; and the nonlinearity h is a real analytic function of the form $h(u) = u^3 + \mathcal{O}(u^4)$.

The technology of the Birkhoff normal forms has been widely used in the study of the dynamics of Hamiltonian systems close to elliptic equilibrium points. For example, obtaining Birkhoff normal forms of the Hamiltonians is the most important step of the KAM approach, which is one of the main tools to deal with the existence of periodic and quasi-periodic solutions of nonlinear PDEs.

This paper is devoted to transform the Hamiltonians of a kind of wave equations to the four-order Birkhoff normal forms. This kind of systems contains nonlinear terms with quasi-periodically forcing and the space variable. We obtain a quantitative description about the Hamiltonian's proposition in a ball of a Sobolev type phase space. The result in this paper provides a basis for the forthcoming research of the existence of periodic or quasi-periodic solutions. The method in this paper can be considered as an idea to deal with the infinite-dimensional systems whose nonlinear terms depend on the time or space variables.

For the Birkhoff normal forms of wave equations under Dirichlet boundary conditions, the reader is referred to [1-4]. However, the partial differential equations with periodic boundary conditions are more complicated since the eigenvalues are not distinct but multiple. This fact would bring a lot of trouble in constructing normal forms. The reason mainly lies in the notorious "small divisor problem", which makes it difficult to obtain the regularity of the symplectic transformations. In [5], the author studied the completely resonant nonlinear wave equation under periodic boundary conditions. But the difficulty caused by the multiplicity of eigenvalues was avoided since the author only considered the even solutions. Articles [6] and [7] succeeded in constructing Birkhoff normal forms of wave equations with periodic boundary conditions and proved that the existence of quasi-periodic solutions. However, their results cannot be used in equations with constant potential.

In this paper, we are interested in the nonlinear wave equations with constant potential and with the nonlinear terms depending on time or space variables. In fact, Berti and Procesi [8] considered the periodically forced wave equations: $\begin{cases} v_{tt} - v_{xx} + f(\omega_1 t, v) = 0 \\ v(t,x) = v(t, x+2\pi), \end{cases}$ with the nonlinear forcing term: $f(\omega_1 t, v) = a(\omega_1 t) v^{2d-1} + \mathcal{O}(v^{2d})$, $d > 1$, $d \in \mathbb{N}_+$ being $2\pi/\omega_1$ -periodic in time t.

Zhang and Si [9] focused on the quasi-periodically forced nonlinear wave equations:

* ***Corresponding author*** e-mail: yiwang8080@126.com

$u_{tt}-u_{xx}+\mu u+\varepsilon\phi(t)h(u)=0,\quad \mu>0$ with Dirichlet boundary conditions, where ϕ is real analytic quasi-periodic function and $h(u)=\eta_1 u+\eta_{2r+1}u^{2r+1}+\sum_{k\geq r+1}\eta_{2k+1}u^{2k+1}$, $\eta_1,\eta_{2r+1}\neq 0,\quad r\in\mathbb{N}$.

In the above equations, one needs to deal with essentially finite small divisors. Moreover, the above equations exclude those cases where the nonlinear terms contain the space variable, while in this paper; we provide an idea to deal with those cases. Factually, in those cases, the important "compactness property" cannot hold. Thus, one would confront essentially infinite small divisors. To overcome this point, we truncate the unperturbed term as well as the perturbed term. Therefore, although the "compactness property" is not satisfied, we can also estimate the measure of the small divisors. Our main Theorem 3.1 proves that there is a canonical transformation, which can change the Hamiltonian to a four-order Birkhoff normal form.

The paper is organized as follows. In section 2, we will give the expression of Hamiltonian. Section 3 is devoted to the Birkhoff normal form of the Hamiltonian.

2 Hamiltonian setting

Throughout this paper, we assume that:

(H) $\quad g_0:=\lim_{T\to\infty}\frac{1}{T}\int_0^T g(\omega t,x)dt\equiv const.\quad 0\neq g_0\in\mathbb{R}.$

For $f\equiv 0$, the equation (1.1) becomes:

$$u_{tt}-u_{xx}+\mu u=0. \tag{2.1}$$

The operator $A=-\frac{d^2}{dx^2}+\mu$ with periodic boundary conditions admits a complete orthogonal basis of eigenfunctions $\phi_j\in L^2([0,2\pi])$, $j\in\mathbb{Z}$, with corresponding eigenvalues $\zeta_j=j^2+\mu$, if one sets $\phi_0=1/\sqrt{2\pi}$ and for $j\geq 1$, $\phi_j(x)=\frac{1}{\sqrt{\pi}}\cos(jx)$, $\phi_{-j}(x)=\frac{1}{\sqrt{\pi}}\sin(jx)$.

Every solution of the linear wave equation (2.1) can be written as a super-position of the basic modes ϕ_j, namely, for $\mathcal{I}$ any subset of $\mathbb{Z}$ and $\mu_j:=\sqrt{\zeta_j}$, $u(x,t)=\sum_{j\in\mathcal{I}}\xi_j\cos(\mu_j t+\theta_j)\phi_j(x)$, with amplitudes $\xi_j>0$ and initial phases θ_j.

In the whole of this paper, we denote by C the universal constants if we do not care their values. For some $\sigma_1>0$ and $\sigma>0$, we suppose that g analytically in ϑ,x extends to the domain $D_1(\sigma_1)\times D(\sigma)$, where $D_1(\sigma_1)=\{\vartheta\,|\,|\mathrm{Im}\vartheta|<\sigma_1\}$ and $D(\sigma)=\{x\,|\,|\mathrm{Im}x|<\sigma\}$.

We rewrite the wave equation (1.1) as follows:

$$\dot{u}=v,\quad \dot{v}+Au=-\varepsilon g(\omega t,x)h(u), \tag{2.2}$$

where $A=-d^2/dx^2+\mu, t\in\mathbb{R}$. As is well known, the equation (2.2) can be studied as an infinite dimensional Hamiltonian system by taking the phase space to be product of the Sobolev spaces $H_0^1([0,2\pi])\times L^2([0,2\pi])$ with coordinates u and $v=\partial_t u$. The Hamiltonian for (2.2) is then $H=\frac{1}{2}(v,v)+\frac{1}{2}(Au,u)+\varepsilon\int_{\mathbb{T}}\chi(u,x,\omega t)dx$, where $\chi(u,x,\omega t)=g(\omega t,x)[\frac{1}{4}u^4+\mathcal{O}(u^5)]$, and $(\cdot,\cdot)$ denotes the usual scalar product in $L^2([0,2\pi])$.

We introduce the coordinates $q=(q_0,q_1,q_{-1},\dots)$ and $p=(p_0,p_1,p_{-1},\dots)$ by setting $u(t,x)=\sum_{j\in\mathbb{Z}}q_j(t)\phi_j(x)$, $v=\sum_{j\in\mathbb{Z}}p_j(t)\phi_j(x)$.

The coordinates are taken from some Banach space $l_b^s(s>0)$ of all real valued bi-infinite sequences $q=(q_0,q_1,q_{-1},\dots)$ with finite norm $\|q\|_s=\sum_{j\in\mathbb{Z}}(j)^s|q_j|$, where $(j)=\max(1,|j|)$. We can obtain the Hamiltonian: $H=\Lambda+G$, where

$$\Lambda=\frac{1}{2}\sum_j(\mu_j^2q_j^2+p_j^2),\quad G=\varepsilon\int_{\mathbb{T}}\chi\left(\sum_{j\in\mathbb{Z}}q_j(t)\phi_j(x),x,\omega t\right)dx$$

and $\mu_j=\sqrt{\zeta_j}$.

The equations of motion are: $\dot{q}_j=\frac{\partial H}{\partial p_j}=p_j$, $\dot{p}_j=-\frac{\partial H}{\partial q_j}=-\mu_j^2q_j-\frac{\partial G}{\partial q_j}$ with respect to the symplectic structure $\sum dp_i\wedge dq_i$ on $l_b^s\times l_b^s$.

To make the system turn into an autonomous system, we introduce a pair of action-angle variables $(J,\vartheta)\in\mathbb{R}^m\times\mathbb{T}^m$ $(\mathbb{T}^m:=\mathbb{R}^m/2\pi\mathbb{Z}^m)$ by assuming that $\vartheta=\omega t$. Then, $\dot{q}_j=\frac{\partial H}{\partial p_j}$, $\dot{p}_j=-\frac{\partial H}{\partial q_j}$, $\dot{\vartheta}=\omega$, $\dot{J}=-\frac{\partial G}{\partial\vartheta}=-\varepsilon\frac{\partial\int_{\mathbb{T}}\chi dx}{\partial\vartheta}$ can be written as a Hamiltonian system (with respect to the symplectic structure $(d\vartheta\wedge dJ+\sum dp_i\wedge dq_i)$ with the Hamiltonian:

$$H=<\omega,J>+\frac{1}{2}\sum_j(\mu_j^2q_j^2+p_j^2)+G(q,\vartheta). \tag{2.3}$$

To continue our investigation of the Hamiltonian (2.3), we need to establish the regularity of the nonlinear Hamiltonian vector field X_G associated to G, where $<\cdot,\cdot>$ is the standard inner product in $\mathbb{C}^m$.

To this end, let l_b^2 and L^2, respectively, be the Hilbert spaces of all bi-infinite, square summated sequences with complex coefficients and all square-integrable complex-valued functions on $[0,2\pi]$. Let $\mathcal{F}: l_b^2 \to L^2$, $q \mapsto \mathcal{F}q = \sqrt{\frac{1}{\pi}}\sum_j q_j e^{\mathrm{i}jx}$ be the inverse discrete Fourier transform, which defines an isometry between the two spaces. Let $s \ge 1$. The subspaces $l_b^s \subset l_b^2$ consist, by definition of all bi-infinite sequences with the finite form $\|q\|_s = \sum_j (j)^s |q_j|$. Through $\mathcal{F}$ we define subspaces $H^s[0,2\pi] \subset L^2[0,2\pi]$ that are normalized by setting $\|\mathcal{F}q\|_s = \|q\|_s$.

The following lemma was proved in [7], we only give the result.

Lemma 2.1 For all $s > 0$, the space l_b^s is a Banach algebra with respect to convolution of the sequences $(q * p)_j := \sum_k q_{j-k} p_k$, and $\|p * q\|_s \le 2^s \|q\|_s \|p\|_s$.

Using the above lemma, we can prove the following lemma.

Lemma 2.2 For all $s \ge 1$, the gradient $\partial_q G$ is real analytic as a map from some neighbourhood of origin in $l_b^s \to l_b^s$, with $\|\partial_q G\|_s = \varepsilon\mathcal{O}(\|q\|_s^3)$.

Proof Let $q \in l_b^s$. Consider as a function on $[0,2\pi]$, $u = \sum q_j \phi_j$ is in H^s with $\|u\|_s \le \|q\|_s$. From Assumption **(H)**, we assume

$$g(\vartheta,x) = g_0 + \sum_{|k|\ge 1}\Big[\sum\nolimits'_{\tau} g_k^{\tau} e^{\mathrm{i}\tau x}\Big] e^{\mathrm{i}<k,\vartheta>}$$

$$= \sqrt{\frac{1}{\pi}}\left\{\sqrt{\pi} g_0 + \sum\nolimits'_{\tau}\sqrt{\pi}\Big[\sum_{|k|\ge 1} g_k^{\tau} e^{\mathrm{i}<k,\vartheta>}\Big] e^{\mathrm{i}\tau x}\right\},$$

where the prime symbol in the summation sign indicates that the sum runs over all $\tau \in \mathbb{Z}$. By using of Lemma A.1 in [10], $|g_k^{\tau}| \le \|g(\vartheta,x)\|_{D(\sigma_1)\times D(\sigma)} e^{-|k|\sigma_1} e^{-|\tau|\sigma}$.

Furthermore, for $\vartheta \in D(\frac{\sigma_1}{2})$,

$$\Big|\sum_{|k|\ge 1} g_k^{\tau} e^{\mathrm{i}<k,\vartheta>}\Big| \le \|g(\vartheta,x)\|_{D(\sigma_1)\times D(\sigma)} e^{-|\tau|\sigma} \sum_{|k|\ge 1} e^{-|k|\sigma_1} e^{|k|\frac{\sigma_1}{2}}$$
$$\le C\|g(\vartheta,x)\|_{D(\sigma_1)\times D(\sigma)} e^{-|\tau|\sigma},$$

because of the convergence of the series $\sum_{|k|\ge 1} e^{-|k|\frac{\sigma_1}{2}}$.

Hence, for $(\vartheta,x) \in D(\frac{\sigma_1}{2}) \times D(\frac{\sigma}{2})$,

$$\|g(\vartheta,x)\|_s = |\sqrt{\pi} g_0| + C\sqrt{\pi}\|g(\vartheta,x)\|_{D(\sigma_1)\times D(\sigma)} \cdot \sum_{|\tau|\ge 0} (\tau)^s e^{-|\tau|\sigma} e^{|\tau|\frac{\sigma}{2}} \le C, \tag{2.4}$$

because of the convergence of the series $\sum_{|\tau|\ge 0} (\tau)^s e^{-|k|\frac{\sigma}{2}}$, where C depends on g, σ_1, s and σ. That is $g(\cdot,\cdot) \in H^s[0,2\pi]$. By the algebra property and the analyticity of g and h from (2.4), the function $g(\vartheta,x)h(u)$ also belongs to $H^s[0,2\pi]$ with $\|g(\vartheta,x)h(u)\|_s \le C\|q\|_s^3$ in a sufficiently small neighbourhood of the origin, where C depends on s, σ, σ_1 and g. On the other hand, since $\frac{\partial G}{\partial q_j} = \varepsilon\int_{\mathbb{T}} g(\vartheta,x)h(u)\phi_j(x)dx$.

The components of G_q are the Fourier coefficients of $g(\vartheta,x)h(u)$, so G_q belongs to l_b^s, with $\|G_q\|_s \le \varepsilon \cdot \|g(\vartheta,x)h(u)\|_s \le \varepsilon C\|q\|_s^3$. The regularity of G_q follows from the regularity of its component and its local boundedness.

3 Partial Birkhoff normal forms

Since $\chi(u,x,\vartheta) = g(\vartheta,x)[\frac{1}{4}u^4 + \mathcal{O}(u^5)]$ and $u = \sum_j q_j \phi_j$, we find that

$$G(q,\vartheta) = \frac{\varepsilon}{4}\sum_{i,j,d,l}\int_{\mathbb{T}} g(\vartheta,x)\phi_i\phi_j\phi_d\phi_l dx q_i q_j q_d q_l + \varepsilon\mathcal{O}(\|q\|_s^5).$$

From (**H**), we can get that

$$g(\vartheta,x) = g_0 + \sum_{|k|\ge 1} g_k(x) e^{\mathrm{i}<k,\vartheta>}. \tag{3.1}$$

It follows from (3.1) that

$$G(q,\vartheta) = \frac{\varepsilon}{4}\sum_{i,j,d,l} G_{ijdl} q_i q_j q_d q_l + \frac{\varepsilon}{4}\sum_{|k|\ge 1,i,j,d,l} G_{k,ijdl} e^{\mathrm{i}<k,\vartheta>} q_i q_j q_d q_l + \varepsilon\mathcal{O}(\|q\|_s^5),$$

where

$$G_{ijdl} = g_0\int_{\mathbb{T}}\phi_i\phi_j\phi_d\phi_l dx \quad \text{and} \quad G_{k,ijdl} = \int_{\mathbb{T}} g_k(x)\phi_i\phi_j\phi_d\phi_l dx, \quad |k| \ge 1. \tag{3.2}$$

An easy computation shows that $G_{ijdl} = 0$ unless $i \pm j \pm d \pm l = 0$ for at least one combination of plus and minus signs. In particular, we have

$$G_{ijij}=\begin{cases}\frac{g_0(2+\delta_{ij})}{4\pi}, & if \quad ij>0\\ \frac{g_0(2-\delta_{i(-j)})}{4\pi}, & if \quad ij<0\\ \frac{g_0}{2\pi}, & if \quad ij=0,\end{cases}$$ where

$\delta_{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j.\end{cases}$ This will play an important role later on.

Given a fixed finite subset of indices $\mathcal{I}_N=\{n_1,\ldots,n_N\}\subset\mathbb{Z}$ with $|n_i|\neq|n_j|$, if $i\neq j$, we decompose the Hamiltonian (2.3) as $H=H_N+H_\infty$, where

$$H_N=\Lambda_N+\varepsilon G_N, \tag{3.3}$$

$$H_\infty=\Lambda_\infty+\varepsilon G_\infty, \qquad \Lambda_N=<\omega,J>+\frac{1}{2}\sum_{j\in\mathcal{I}_N}(\mu_j^2q_j^2+p_j^2),$$

$$G_N(q,\vartheta)=\frac{1}{4}\sum_{i,j,d,l\in\mathcal{L}_N,i\pm j\pm d\pm l=0}G_{ijdl}q_iq_jq_dq_l$$
$$+\frac{1}{4}\sum_{|k|\geq1,i,j,d,l\in\mathcal{L}_N}G_{k,ijdl}e^{i<k,\vartheta>}q_iq_jq_dq_l+\mathcal{O}(|q|^5), \tag{3.4}$$

$$\Lambda_\infty=\frac{1}{2}\sum_{j\notin\mathcal{I}_N}(\mu_j^2q_j^2+p_j^2),$$
$$and \quad G_\infty=G(q,\vartheta)-G_N(q,\vartheta), \tag{3.5}$$

where $\mathcal{O}(|q|^5)$ denotes the five order terms, in which all the subscripts of q belong to the subset $\mathcal{I}_N$ and noticing that N is finite.

We introduce the complex coordinates z_j, $j=1,2,\ldots,N$ by $z_j=\frac{1}{\sqrt{2\mu_{n_j}}}(\mu_{n_j}q_{n_j}+\mathrm{i}p_{n_j})$ and define $|z|^2=|z_1|^2+\ldots+|z_N|^2$ for a vector $z=(z_1,\ldots,z_N)$. So we obtains the Hamiltonian:

$$H_N(z,\bar{z})=\Lambda_N+\varepsilon G_N=$$
$$<\omega,J>+\sum_{j=1,2,\ldots,N}\mu_{n_j}|z_j|^2+\varepsilon G_N(z,\bar{z},\vartheta), \tag{3.6}$$

with symplectic structure $d\vartheta\wedge dJ+\mathrm{i}\sum_j dz_j\wedge\bar{z}_j$, where

$$G_N(\bar{z},z,\vartheta)=\frac{1}{4}\sum_{n_i\pm n_j\pm n_d\pm n_l=0,\,i,j,d,l=1,\ldots,N}G_{n_in_jn_dn_l}$$
$$\cdot\frac{(z_i+\bar{z}_i)}{\sqrt{2\mu_{n_i}}}\frac{(z_j+\bar{z}_j)}{\sqrt{2\mu_{n_j}}}\frac{(z_d+\bar{z}_d)}{\sqrt{2\mu_{n_d}}}\frac{(z_l+\bar{z}_l)}{\sqrt{2\mu_{n_l}}}$$
$$+\frac{1}{4}\sum_{|k|\geq1}\sum_{i,j,d,l=1,\ldots,N}G_{k,n_in_jn_dn_l}e^{\mathrm{i}<k,\vartheta>}$$
$$\cdot\frac{(z_i+\bar{z}_i)}{\sqrt{2\mu_{n_i}}}\frac{(z_j+\bar{z}_j)}{\sqrt{2\mu_{n_j}}}\frac{(z_d+\bar{z}_d)}{\sqrt{2\mu_{n_d}}}\frac{(z_l+\bar{z}_l)}{\sqrt{2\mu_{n_l}}}+\mathcal{O}(|z|^5) \tag{3.7}$$

By using the method in [7], for the remaining coordinates, one introduces the notation, for $\nu\geq1$,

$$x'_\nu=\begin{cases}(q_\nu,q_{-\nu})\in\mathbb{R}^2, & \text{if} \quad \nu,-\nu\notin\mathcal{I}_N,\\ q_{-\tilde{\nu}}\in\mathbb{R}, & \text{if} \quad \nu=|\tilde{\nu}| \text{ for some } \tilde{\nu}\in\mathcal{I}_N,\end{cases}$$

and similarly for $p_j, j\notin\mathcal{I}_N$, denoted in term of $y_\nu\in\mathbb{R}^{N_\nu},\nu\geq1$, with N_ν as above, namely, $N_\nu=2$ if both $\nu,-\nu\notin\mathcal{I}_N$ and $N_\nu=1$ otherwise. For d_k, $k\geq1$, a sequence of strictly positive integers uniformly bounded by some $\bar{d}<\infty$, let $\mathcal{R}^\infty$ denote the set of infinite sequences $x'=(x'_1,x'_2,\ldots)$ with $x'_k\in\mathbb{R}^{d_k}$. Then we can introduce the following family of Banach spaces $\mathcal{R}_s^\infty$, $s\in\mathbb{R}$, $\mathcal{R}_s^\infty=\{Z\in\mathcal{R}^\infty \,\|\, Z|\equiv\sum_{k\geq1}k^s|Z_k|_{\mathbb{R}^{d_k}}\}$. Clearly, for $q,p\in l_b^s$ one has $x',y\in\mathcal{R}_s^\infty$, and $H_\infty(z,\bar{z},p,q)$ reads in these notations

$$H_\infty(z,\bar{z},x',y,\vartheta)=\Lambda_\infty(x',y)+\varepsilon G_\infty(z,\bar{z},x',\vartheta),$$

$$\Lambda_\infty(x',y)=\frac{1}{2}\sum_{\nu\geq1}(\mu_\nu^2|x'_\nu|^2+|y_\nu|^2), \tag{3.8}$$

and $|G_\infty|=\mathcal{O}(\sum_{l=0}^{3}|z|^l\,\|x'\|_s^{4-l})+\mathcal{O}(\sum_{l=0}^{4}|z|^l\,\|x'\|_s^{5-l})$.

Theorem 3.1 Choose ϵ_0 small enough. Consider the Hamiltonian H_N. For each fixed subset $\mathcal{I}_N,N<\infty$, satisfying $|n_i|\neq|n_j|$ when $i\neq j$, there is a subset $\Omega\subset[\varrho,2\varrho]^m$ with $meas\Omega>0$ such that for any $\omega\in\Omega$, and there is a real analytic, symplectic change of coordinates Ψ_N in a complex neighbourhood: $\vartheta\in D(\frac{\sigma_1}{2}):=\{\vartheta\,|\,|\mathrm{Im}\vartheta|<\frac{\sigma_1}{2},\sigma_1>0\}$ of the tour $\mathbb{T}^m$ and a neighbourhood of the origin in $\mathbb{C}^N$ such that for all $\mu>0$, the Hamiltonian (3.3) can be transformed into $H_N\circ\Psi_N=\Lambda_N+\varepsilon\bar{G}_N+\varepsilon\hat{G}_N+\varepsilon^2K_N+\varepsilon\mathcal{O}(|z|^5)$, where $\hat{G}_N=\epsilon_0\mathcal{O}(|z|^4),K_N=\mathcal{O}(|z|^6)$, and $\bar{G}_N(z,\bar{z})=\frac{1}{2}\sum_{i,j=1}^{N}\bar{g}_{ij}|z_i|^2|z_j|^2$ with uniquely determined coefficient

$$\bar{g}_{ij}=\begin{cases}\dfrac{3g_0}{4\pi\mu_{n_{|i|}}\mu_{n_{|j|}}}, & \text{if } i\neq j;\\ \dfrac{9g_0}{16\pi\mu_{n_{|i|}}\mu_{n_{|i|}}}, & \text{if } i=j \text{ and } n_{|i|}=n_{|j|}\neq 0;\\ \dfrac{3g_0}{8\pi\mu_{n_{|i|}}\mu_{n_{|i|}}}, & \text{if } i=j \text{ and } n_{|i|}=n_{|j|}=0.\end{cases}$$

Furthermore, setting $\Psi_\infty=\Psi_N\oplus 1_{\mathcal{R}_s^\infty\times\mathcal{R}_s^\infty}$, one has $H_\infty\circ\Psi_\infty=\Lambda_\infty+\varepsilon K_\infty$ with

$$|K_\infty|=\mathcal{O}(\sum_{l=0}^{3}|z|^l\,\|x'\|_s^{4-l})+\mathcal{O}(\sum_{l=0}^{4}|z|^l\,\|x'\|_s^{5-l}), \quad (3.9)$$

where 1 denotes the identity map.

Before the proof of the above theorem, we first prove the following lemmas.

Lemma 3.1 There is a set $\underline{\Omega}\subset[\varrho,2\varrho]^m$ $(\varrho>0)$ such that for any $\omega\in\underline{\Omega}$ satisfying that

$$|<k,\omega>|\geq\frac{\varrho\varepsilon}{|k|^{m+1}},\quad for\ \ all\ \ 0\neq k\in\mathbb{Z}^m \quad (3.10)$$

And $meas\underline{\Omega}\geq(1-C_1\varepsilon)\varrho^m$, where the constant C_1 depends on m.

Proof Let $0\neq k\in\mathbb{Z}^m$, $\mathcal{R}_k^1=\left\{\omega\in[\varrho,2\varrho]^m:|<k,\omega>|<\dfrac{\varrho\varepsilon}{|k|^{m+1}}\right\}$, and $\mathcal{R}^1=\bigcup_{0\neq k\in\mathbb{Z}^m}\mathcal{R}_k^1$. Consider two hyperplanes $<k,\omega>=\pm\dfrac{\varrho\varepsilon}{|k|^{m+1}}$. We have

$$\text{meas}\mathcal{R}_k^1\leq m|k|^{-1}(\sqrt{2}\varrho)^{m-1}\frac{2\varrho\varepsilon}{|k|^{m+1}}\leq\frac{2(\sqrt{2})^{m-1}m\varepsilon}{|k|^{m+2}}\varrho^m.$$

It follows that

$$\text{meas}\mathcal{R}^1\leq\sum_{0\neq k\in\mathbb{Z}^m}\text{meas}\mathcal{R}_k^1\leq 2(\sqrt{2})^{m-1}m\varepsilon\varrho^m\sum_{0\neq k\in\mathbb{Z}^m}\frac{1}{|k|^{m+2}}$$
$$\leq C_1\varepsilon\varrho^m\sum_{p=1}^{\infty}(2p+1)^{m-1}p^{-(m+2)}\leq C_1\varepsilon\varrho^m,$$

because the series $\sum_{p=1}^{\infty}(2p+1)^{m-1}p^{-(m+2)}$ is convergent. Therefore, this lemma is true when we assume that $\underline{\Omega}=[\varrho,2\varrho]^m\setminus\mathcal{R}^1$.

Now, we use the notation $\mu_{i'}=\text{sgn}i\cdot\mu_{n_{|i|}}$.

Lemma 3.2 Assume that $n_{|i|},n_{|j|},n_{|d|},n_{|l|}\in\mathcal{I}_N$ are integers, $i,j,d,l\in\{1,-1,2,-2,...,N,-N\}$ and $1\leq|k|\leq K_0$, where we choose $K_0>K_0':=\dfrac{4}{\sigma_1}ln(\epsilon_0^{-1})$.

Then, for the parameter set $[\varrho,2\varrho]^m$, there is a subset $\bar{\Omega}\subset[\varrho,2\varrho]^m$ with

$$meas\bar{\Omega}\geq\varrho^m\left(1-\frac{C_2\varepsilon}{\ln(\epsilon_0^{-1})}\right), \quad (3.11)$$

satisfying that, for any $\omega\in\bar{\Omega}$,

$$\left|\mu_i'+\mu_j'+\mu_d'+\mu_l'+<k,\omega>\right|\geq\frac{\varrho\varepsilon}{K_0^{m+1}}, \quad (3.12)$$

where C_2 is a constant depending on N, m, and σ_1.

Proof Assume $\mathcal{R}_{ijdl,k}^2=\{\omega\in[\varrho,2\varrho]^m:|\mu_i'+\mu_j'+\mu_d'+\mu_l' +<k,\omega>|<\dfrac{\varrho\varepsilon}{K_0^{m+1}}\}$ and $\Omega^2=\bigcup_{1\leq|k|\leq K_0}\bigcup_{i,j,d,l}\mathcal{R}_{ijdl,k}^2$. It follows that, by using of the same method in the proof of Lemma 3.1, for fixed i,j,d,l and k,

$$\text{meas}\mathcal{R}_{ijdl,k}^2\leq C\frac{\varrho^m\varepsilon}{K_0^{m+1}|k|}\leq C\frac{\varrho^m\varepsilon}{K_0^{m+1}}, \quad (3.13)$$

where C depends on *m.* It is well known that the number

$$\sharp\{k\in\mathbb{Z}^m:|k|=l\}\leq 2^m l^{m-1}. \quad (3.14)$$

So

$$\sharp\{k\in\mathbb{Z}^m:1\leq|k|\leq K_0\}\leq 2^m\sum_{l=1}^{K_0}l^{m-1}\leq 2^m K_0^m. \quad (3.15)$$

It yields that, from (3.13), $\text{meas}\Omega^2=\text{meas}\bigcup_{1\leq|k|\leq K_0}\bigcup_{1\leq|i|,|j|,|d|,|l|\leq N}\mathcal{R}_{ijdl,k}^2$

$$\leq\frac{C\varrho^m\varepsilon}{K_0^{m+1}}(2N)^4 2^m K_0^m\leq\frac{C_2\varrho^m\varepsilon}{K_0}\leq C_2\frac{\varrho^m\varepsilon}{\ln(\epsilon_0^{-1})},$$

where C_2 is a constant depending on N, σ_1, and m. Finally, we only need to assume $\bar{\Omega}=[\varrho,2\varrho]^m/\Omega^2$. This completes the proof.

Now we prove Theorem 3.1.

Proof of Theorem 3.1 We always suppose, that $n_{|i|}, n_{|j|}, n_{|d|}, n_{|l|} \in \mathcal{I}_N$. It is convenient to adopt the notation $z_j = w_j$, $\bar{z}_j = w_{-j}$, $j = 1, 2, ..., N$, in which H_N reads, from (3.6) and (3.7),

$$H_N = \Lambda_N + \varepsilon G_N$$
$$=< \omega, J > + \sum_{j=1}^{N} \mu_{n_j} w_j w_{-j} + \frac{\varepsilon}{16} \sum_{i,j,d,l,\, n_{|i|} \pm n_{|j|} \pm n_{|d|} \pm n_{|l|} = 0} {}' g_{ijdl}$$
$$\cdot w_i w_j w_d w_l + \frac{\varepsilon}{16} \sum_{|k| \ge 1} \sum_{i,j,d,l} {}' g_{k,ijdl} e^{\mathrm{i} < k, \vartheta >} w_i w_j w_d w_l + \varepsilon \mathcal{O}(|w|^5),$$

where

$$g_{ijdl} = \frac{G_{n_{|i|} n_{|j|} n_{|d|} n_{|l|}}}{\sqrt{\mu_{n_{|i|}} \mu_{n_{|j|}} \mu_{n_{|d|}} \mu_{n_{|l|}}}}, \qquad g_{k,ijdl} = \frac{G_{k, n_{|i|} n_{|j|} n_{|d|} n_{|l|}}}{\sqrt{\mu_{n_{|i|}} \mu_{n_{|j|}} \mu_{n_{|d|}} \mu_{n_{|l|}}}}, \tag{3.16}$$

and the prime symbol in the summation sign indicates that the sum runs over all indices $i, j, d, l \in \{1, -1, ..., N, -N\}$.

Consider a Hamiltonian function $\mathcal{F} = \varepsilon F = \varepsilon \sum_{i,j,d,l} {}' F_{ijdl} w_i w_j w_d w_l + \varepsilon \sum_{0 < |k| \le K_0} \sum_{i,j,d,l} {}' F_{k,ijdl} e^{\mathrm{i} < k, \vartheta >} w_i w_j w_d w_l$, where we define, for $k = 0$, $\mathrm{i} F_{ijdl} = \frac{g_{ijdl}}{16(\mu_i' + \mu_j' + \mu_d' + \mu_l')}$, if $\mu_i', \mu_j', \mu_d', \mu_l' \not\equiv \{a, -a, b, -b\}$ and $n_{|i|} \pm n_{|j|} \pm n_{|d|} \pm n_{|l|} = 0$ and $\mathrm{i} F_{ijdl} = 0$, if $\mu_i', \mu_j', \mu_d', \mu_l' \equiv \{a, -a, b, -b\}$ and $n_{|i|} \pm n_{|j|} \pm n_{|d|} \pm n_{|l|} = 0$, or $n_{|i|} \pm n_{|j|} \pm n_{|d|} \pm n_{|l|} \ne 0$; for $k \ne 0$,

$$\mathrm{i} F_{k,ijdl} = \begin{cases} \dfrac{g_{k,ijdl}}{16 < k, \omega >}, & \text{if } \mu_i' + \mu_j' + \mu_d' + \mu_l' = 0, \\ \dfrac{g_{k,ijdl}}{16(\mu_i' + \mu_j' + \mu_d' + \mu_l' + < k, \omega >)}, & \textit{otherwise}. \end{cases}$$

In the same way with [3] and [7], we can prove that, for the integers $n_{|i|}, n_{|j|}, n_{|d|}, n_{|l|} \in \mathcal{I}_N$ satisfying $n_{|i|} \pm n_{|j|} \pm n_{|d|} \pm n_{|l|} = 0$, the following inequality

$$|\mu_i' + \mu_j' + \mu_d' + \mu_l'| \ge \frac{c\mu}{(M^2 + \mu)^{3/2}} > 0, \tag{3.17}$$

holds, where c is some absolute constant and $M = \min \{n_{|i|}, n_{|j|}, n_{|d|}, n_{|l|}\}$.

Let $\Psi_N = X_{\mathcal{F}}^1$ be the time-1 map of the vector-field of the Hamiltonian $\mathcal{F}$. Expanding at $t = 0$ and using Taylor's formula we can obtain that

$$H_N \circ \Psi_N = H_N + \{H_N, \mathcal{F}\} + \int_0^1 (1 - t)\{\{H_N, \mathcal{F}\}, \mathcal{F}\} \circ X_{\mathcal{F}}^t dt$$
$$= \Lambda_N + \varepsilon \tilde{G}_N + \varepsilon \{\Lambda_N, F\} + \varepsilon \hat{G}_N + \varepsilon^2 \{G_N, F\}$$
$$+ \varepsilon^2 \int_0^1 (1 - t)\{\{H_N, F\}, F\} \circ X_{\mathcal{F}}^t dt,$$

where

$$\tilde{G}_N(w, \vartheta) = G_N(w, \vartheta) - \hat{G}_N(w, \vartheta)$$
$$= \frac{1}{16} \sum_{i,j,d,l,\, n_{|i|} \pm n_{|j|} \pm n_{|d|} \pm n_{|l|} = 0} {}' g_{ijdl} w_i w_j w_d w_l$$
$$+ \frac{1}{16} \sum_{0 < |k| \le K_0} \sum_{i,j,d,l} {}' g_{k,ijdl} e^{\mathrm{i} < k, \vartheta >} w_i w_j w_d w_l + \mathcal{O}(|w|^5),$$

and $\{\cdot, \cdot\}$ is the Poisson bracket of smooth functions:

$$\{G_1, G_2\} = \frac{\partial G_1}{\partial \vartheta} \frac{\partial G_2}{\partial J} - \frac{\partial G_1}{\partial J} \frac{\partial G_2}{\partial \vartheta} + \mathrm{i} \sum_{j=1}^{N} \left(\frac{\partial G_1}{\partial z_j} \frac{\partial G_2}{\partial \bar{z}_j} - \frac{\partial G_1}{\partial \bar{z}_j} \frac{\partial G_2}{\partial z_j} \right).$$

Now let us compute $\{\Lambda_N, F\}$:

$$\{\Lambda_N, F\} =$$
$$-\mathrm{i} \sum_{i,j,d,l,\, n_{|i|} \pm n_{|j|} \pm n_{|d|} \pm n_{|l|} = 0} {}' (\mu_i' + \mu_j' + \mu_d' + \mu_l') F_{ijdl} w_i w_j w_d w_l$$
$$-\mathrm{i} \sum_{0 < |k| \le K_0} \sum_{i,j,d,l} {}' (\mu_i' + \mu_j' + \mu_d' + \mu_l' + < k, \omega >)$$
$$\cdot F_{k,ijdl} e^{\mathrm{i} < k, \vartheta >} w_i w_j w_d w_l$$

Hence

$$\tilde{G}_N + \{\Lambda_N, F\}$$
$$= \sum_{i,j,d,l,\, n_{|i|} \pm n_{|j|} \pm n_{|d|} \pm n_{|l|} = 0} {}' \left(\frac{1}{16} g_{ijdl} - \mathrm{i}(\mu_i' + \mu_j' + \mu_d' + \mu_l') F_{ijdl} \right)$$
$$\cdot w_i w_j w_d w_l$$
$$+ \sum_{i,j,d,l} {}' \sum_{0 < |k| \le K_0} \left(\frac{1}{16} g_{k,ijdl} - \mathrm{i}(\mu_i' + \mu_j' + \mu_d' + \mu_l' + < k, \omega >) \right.$$
$$\left. F_{k,ijdl} \right) \cdot e^{\mathrm{i} < k, \vartheta >} w_i w_j w_d w_l + \mathcal{O}(|w|^5)$$
$$= \bar{G}_N + \mathcal{O}(|w|^5) = \frac{1}{2} \sum_{i,j=1}^{N} \bar{g}_{ij} w_i w_{-i} w_j w_{-j} + \mathcal{O}(|w|^5),$$

where if $i \ne j$, $\bar{g}_{ij} = \frac{24}{16} g_{i-ij-j} = \frac{3g_0}{4\pi \mu_{n_{|i|}} \mu_{n_{|j|}}}$; if $i = j$ and $n_{|i|} = n_{|j|} \ne 0$, $\bar{g}_{ij} = \frac{12}{16} g_{i-ii-i} = \frac{9g_0}{16\pi \mu_{n_{|i|}} \mu_{n_{|i|}}}$; if $i = j$ and $n_{|i|} = n_{|j|} = 0$, $\bar{g}_{ij} = \frac{12}{16} g_{i-ii-i} = \frac{3g_0}{8\pi \mu_{n_{|i|}} \mu_{n_{|i|}}}$.

The uniqueness can be proved in the classical way as same as in [11]. Hence, we have

$$H_N \circ \Psi_N = \Lambda_N + \varepsilon \bar{G}_N + \varepsilon \hat{G}_N + \varepsilon^2 \{G_N, F\}$$
$$+ \varepsilon^2 \int_0^1 (1 - t)\{\{H_N, F\}, F\} \circ X_{\mathcal{F}}^t + \varepsilon \mathcal{O}(|w|^5).$$

CLAIM. The vector-field of the Hamiltonian X_F is real analytic in a complex neighbourhood $\vartheta \in D(\frac{\sigma_1}{2})$ of

$\mathbb{T}^m$ and some neighbourhood of the origin in $\mathbb{C}^N$, and satisfies $|F_w| = \mathcal{O}(|w|^3)$. In fact, letting $w=(w_1,w_2,...,\ w_N)\in\mathbb{C}^N$ $\vartheta\in D(\sigma_1)$, we have that $|w|^2=|w_1|^2+...+\ |w_N|^2$, and from (3.17), (3.16) and (3.2), we have that

$$|F_{\iota jdl}| = \left|\frac{g_{\iota jdl}}{16(\mu_\iota'+\mu_j'+\mu_d'+\mu_l')}\right| \le C|g_{\iota jdl}|$$

$$\le \frac{C}{\mu}|G_{n_{|\iota|}n_{|j|}n_{|d|}n_{|l|}}| \le C|g_0| \le C.$$

Now we let $\Omega=\bar{\Omega}\cap\underline{\Omega}$. By using of Lemmas 3.1 and 3.2, it is obvious, that $meas\Omega\ge\varrho^m(1-C_1\varepsilon-\frac{C_2\varepsilon}{ln(\epsilon_0^{-1})})$. So $meas\Omega>0$ when ε is small enough. In addition, for $(\vartheta,x)\in D(\sigma_1)\times D(\sigma)$, by Lemma A.1 in [10] and from (3.1):

$$|g_k(x)| \le \|g(\vartheta,x)\|_{D(\sigma_1)\times D(\sigma)} e^{-|k|\sigma_1} \tag{3.18}$$

is always true. Therefore, when $\omega\in\Omega$, if $\mu_\iota'+\mu_j'+\mu_d'+\mu_l'=0$, from (3.10), (3.2) and (3.16), we can get

$$|F_{k,\iota jdl}| = \left|\frac{g_{k,\iota jdl}}{16<k,\omega>}\right| \le C\frac{|k|^{m+1}}{\varrho\varepsilon}|g_{k,\iota jdl}|$$

$$\le C\frac{|k|^{m+1}}{\mu\varrho\varepsilon}|G_{k,n_{|\iota|}n_{|j|}n_{|d|}n_{|l|}}| \le C\frac{|k|^{m+1}}{\mu\varrho\varepsilon}\|g(\vartheta,x)\|_{D(\sigma_1)\times D(\sigma)} e^{-|k|\sigma_1}$$

$$\le C|k|^{m+1}\|g(\vartheta,x)\|_{D(\sigma_1)\times D(\sigma)} e^{-|k|\sigma_1}.$$

If $\mu_\iota'+\mu_j'+\mu_d'+\mu_l'\ne0$, we have that, from (3.12), (3.2) and (3.16),

$$|F_{k,\iota jdl}| = \left|\frac{g_{k,\iota jdl}}{16(\mu_\iota'+\mu_j'+\mu_d'+\mu_l'+<k,\omega>)}\right| \le C\frac{K_0^{m+1}}{\varrho\varepsilon}|g_{k,\iota jdl}|$$

$$\le C\frac{K_0^{m+1}}{\mu\varrho\varepsilon}|G_{k,n_{|\iota|}n_{|j|}n_{|d|}n_{|l|}}| \le C\frac{K_0^{m+1}}{\mu\varrho\varepsilon}\|g(\vartheta,x)\|_{D(\sigma_1)\times D(\sigma)} e^{-|k|\sigma_1}$$

$$\le CK_0^{m+1}\|g(\vartheta,x)\|_{D(\sigma_1)\times D(\sigma)} e^{-|k|\sigma_1}.$$

It follows that, by using of (3.15),

$$|F_{w_\iota}| \le 4\sum_{j,d,l,n_{|\iota|}=n_{|j|}\pm n_{|d|}\pm n_{|l|}}{}' |F_{\iota jdl}|\,|w_jw_dw_l| +$$

$$4\sum_{0<|k|\le K_0}\sum_{j,d,l}{}'|F_{k,\iota jdl}e^{i<k,\vartheta>}|\,|w_jw_dw_l|$$

$$\le C\sum_{j,d,l}{}'|w_jw_dw_l| + \sum_{j,d,l}{}'\sum_{0<|k|\le K_0} CK_0^{m+1}\|g(\vartheta,x)\|_{D(\sigma_1)\times D(\sigma)}$$

$$\cdot e^{-|k|\sigma_1}e^{|k|\sigma_1}|w_jw_dw_l| \le C|w|^3 + C\|g(\vartheta,x)\|_{D(\sigma_1)\times D(\sigma)}$$

$$\cdot\sum_{1\le|k|\le K_0} K_0^{m+1}|w|^3 \le C|w|^3 + CK_0^{m+1}2^mK_0^m|w|^3 \le C|w|^3,$$

where C depends on $m,g,N,\sigma_1,\sigma,K_0,\varrho,\varepsilon$ and μ. Therefore, we can get that

$$|F_w| = \sqrt{\sum_{\iota=1}^{N}|F_{w_\iota}|^2} \le C(|w|)^3, \tag{3.19}$$

where C depends on $m,g,N,\sigma_1,K_0,\varrho,\varepsilon,N$, and μ. Similarly, we can prove that, for $(\vartheta,x)\in D(\frac{\sigma_1}{2})\times D(\sigma)$,

$$|(G_N)_{w_\iota}| \le \frac{1}{4}\sum{}'_{j,d,l,n_{|\iota|}\pm n_{|j|}\pm n_{|d|}\pm n_{|l|}=0}|g_{\iota jdl}|\,|w_jw_dw_l|$$

$$+\frac{1}{4}\sum_{|k|\ge1}\sum{}'_{j,d,l}|g_{k,\iota jdl}e^{i<k,\vartheta>}|\,|w_jw_dw_l|+C(|w|^4)$$

$$\le C\sum{}'_{j,d,l,n_{|\iota|}\pm n_{|j|}\pm n_{|d|}\pm n_{|l|}=0}|w_jw_dw_l|+C\sum_{|k|\ge1}\sum{}'_{j,d,l}|g_{k,\iota jdl}|$$

$$e^{|k|\frac{\sigma_1}{2}}|w_jw_dw_l|+C(|w|^4)\le C|w|^3$$

$$+C\sum_{|k|\ge1}\sum{}'_{j,d,l}|G_{k,n_{|\iota|}n_{|j|}n_{|d|}n_{|l|}}|\,e^{|k|\frac{\sigma_1}{2}}|w_jw_dw_l|+C(|w|^4).$$

It follows from (3.18) and (3.14) that, for $|w|\le1$,

$$|(G_N)_{w_\iota}| \le C|w|^3 + C\sum{}'_{j,d,l}\sum_{|k|\ge1}\|g(\vartheta,x)\|_{D(\sigma_1)\times D(\sigma)}$$

$$\cdot e^{-|k|\sigma_1}e^{|k|\frac{\sigma_1}{2}}|w_jw_dw_l|$$

$$\le C|w|^3 + C|w|^3\sum_{|k|\ge1}e^{-|k|\frac{\sigma_1}{2}}$$

$$\le C|w|^3 + C|w|^3\sum_{l\ge1}2^ml^{m-1}e^{-l\frac{\sigma_1}{2}} \le C|w|^3,$$

by using of convergence of series $\sum_{l\ge1}2^ml^{m-1}e^{-l\frac{\sigma_1}{2}}$, where C depends on m,g,N,σ_1,σ, and μ. Thus

$$|(G_N)_w| = \mathcal{O}(|w|^3). \tag{3.20}$$

Suppose that

$K_N=\{G_N,F\}+\int_0^1(1-t)\{\{H_N,F\},F\}\circ X_{\mathcal{F}}^t dt.$

By using of (3.20) and (3.19), we get that

$$|\{G_N,F\}| = \mathcal{O}(|w|^6). \tag{3.21}$$

Using of Cauchy estimates for the fact that $|\{\Lambda_N,F\}|=|\bar{G}_N+\mathcal{O}(|w|^5)-\tilde{G}_N|=\mathcal{O}(|w|^4)$ and from (3.19), it results

$$|\{\{\Lambda_N,F\},F\}| = \mathcal{O}(|w|^6) \tag{3.22}$$

For $\frac{|w|}{2}\le\frac{1}{2}$. Moreover, using Cauchy estimates for (3.21), it derives from (3.19) that $|\{\{G_N,F\},F\}|=\mathcal{O}(|w|^8)$. It follows from (3.22) and (3.3) that $|\{\{H_N, F\}, F\}| = \mathcal{O}(|w|^6)$. Therefore, $|K_N| = \mathcal{O}(|w|^6)$ holds for $\frac{|w|}{2} \le \frac{1}{2}$.

At the end of this proof, we estimate $\hat{G}_N$. By the definition of $g_{k,ijdl}$, from (3.2), (3.14) and (3.16), for $(\vartheta, x) \in D(\frac{\sigma_1}{2}) \times D(\sigma)$ and $K_0 > K_0'$, we have

$$|\sum_{|k|>K_0} g_{k,ijdl} e^{\mathrm{i}<k,\vartheta>}| \le \sum_{|k|>K_0} |g_{k,ijdl}| e^{|k|\frac{\sigma_1}{2}}$$

$$\le \sum_{|k|>K_0} \frac{1}{\mu} |G_{k,n_{|i|}n_{|j|}n_{|d|}n_{|l|}}| e^{|k|\frac{\sigma_1}{2}}$$

$$\le \frac{1}{\mu} \sum_{|k|>K_0} |\int_{\mathbb{T}} g_k(x) \phi_{n_{|i|}} \phi_{n_{|j|}} \phi_{n_{|d|}} \phi_{n_{|l|}} dx| e^{|k|\frac{\sigma_1}{2}}$$

$$\le C \sum_{|k|>K_0} \|g(\vartheta, x)\|_{D(\sigma_1)\times D(\sigma)} e^{-|k|\sigma_1} e^{|k|\frac{\sigma_1}{2}}$$

$$\le C \sum_{|k|>K_0} e^{-|k|\frac{\sigma_1}{2}} \le C \sum_{l>K_0} 2^m l^{m-1} e^{-l\frac{\sigma_1}{2}}$$

$$\le C K_0^m e^{-K_0\frac{\sigma_1}{2}} = C \frac{K_0^m}{e^{K_0\frac{\sigma_1}{4}}} e^{-K_0\frac{\sigma_1}{4}}$$

$$\le C e^{-K_0\frac{\sigma_1}{4}} \le C e^{-\frac{\sigma_1}{4}\frac{4}{\sigma_1}\ln(\epsilon_0^{-1})} = C\epsilon_0,$$

where C depends on g, σ_1, σ and μ, as ϵ_0 small enough. It follows that

$$|\hat{G}_N| = |\frac{1}{16} \sum_{|k|>K_0} \sum{}'_{i,j,d,l} g_{k,ijdl} e^{\mathrm{i}<k,\vartheta>} w_i w_j w_d w_l|$$

$$\le \frac{1}{16} \sum{}'_{i,j,d,l} |\sum_{|k|>K_0} g_{k,ijdl} e^{\mathrm{i}<k,\vartheta>}| |w_i w_j w_d w_l|$$

$$\le C\epsilon_0 \sum{}'_{i,j,d,l} |w_i w_j w_d w_l| \le C\epsilon_0 |w|^4,$$

where C depends on g, σ_1, σ, μ, and N. Hence, $\hat{G}_N = \epsilon_0 \mathcal{O}(|w|^4)$. This completes the proof.

Acknowledgments

This work is partially supported by Program for Soft Science Research Project of Shandong Province (No. 2013RKB01253); this work is also partially supported by NSFC grants (No. 61340043 and No. 61273230).

References

[1] Wayne C E 1990 *Comm. Math. Phys.* **127** 479-528
[2] Poschel J 1996 *Ann. Sc. Norm. Super. Pisa Cl. Sci. IV Ser.* **23** 119-48
[3] Poschel J 1996 *Comm. Math. Helv.* **71** 269-96
[4] Bambusi D 2003 *Comm. Math. Phys.* **234** 253-85
[5] Yuan X 2005 *J. Differential Equations* **230** 213-74
[6] Chierchia L, You J 2000 *Commun. Math. Phys.* **211** 497-525
[7] Bricmont J, Kupiainen A, Schenkel A, 2001 *Commun. Math. Phys.* **211** 101-40
[8] Berti M, Procesi L 2006 *Comm. in Partial Differential Equations* **31** 959-85
[9] Zhang M, Si J 2009 *Physica D* **238** 2185-215
[10] Poschel J 2001 *Proc. Symp. Pure Math.* **69** 707-32
[11] Siegal C L, Moser J K 1971 *Lectures on Celestial Mechanics* vol. Grundlehren **187** Springer: Berlin

Author

Yi Wang, born in May, 1980, Shandong, China

Current position: associate professor of mathematics
University studies: Ph. D from Shandong University in 2012
Scientific interest: differential equations, difference equations, and applications of mathematics in economics
Publications: Yi Wang has published many papers of high scientific quality, including “A result on quasi-periodic solutions of a nonlinear beam equation with a quasi-periodic forcing term” in *Zeitschrift fur Angewandte Mathematik und Physik*, “Quasi-periodic solutions of a quasi-periodically forced nonlinear beam equation” in *Communications in Nonlinear Science and Numerical Simulation*, etc.