A fuzzy set approach for a multi-period optimal portfolio selection model

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Abstract

Due to portfolio decision deals with future events and opportunities, the market information is uncertain. This paper aims to propose a fuzzy multi-period portfolio selection model to hedge against the uncertainty. A new transformation method based on qualitative possibility theory is developed to transfer the model to a crisp programming, which can be solved by an optimization technique. An example is used to illustrate our approach.

Keywords: Multi-period portfolio selection, Fuzzy sets, Optimization, Qualitative possibility theory

1 Introduction

The purpose of an investor is to predict the future return and decide how to allocate asset optimally for Maximilian the total return under some constraints, such as a budget constraints. However, the confront with two problems. One is culti-period invest strategy. Another is uncertainty. As every knows, the time horizon of an investment for an investor usually is multi-period, because he will adjust invest strategy according to market and his budget, from this point, Markovitz's single mean-variance analysis model [1] does not conform to the actual condition. A numerous cholers extended the single period portfolio to the dynamic case. Hakansson [2] gave the multi period mean variance model based on the general portfolio selection theory. Li and Ng [3] generalize Markowitz's mean-variance model to the multi-period model under discrete case, and deduced the analysis expression of efficient frontier by establishing auxiliary function.

It is clear that history data not mean the future data because the market condition is changing time and time. Conventional portfolio optimization models have an assumption that the future condition of stock market can be accurately predicted by historical data. However, no matter how accurate the past data is, this premise will not exist in the financial market due to the high volatility of market environment. To deal with imprecise information in making portfolio selection decisions. Östermark [4] used the fuzzy decision theory to study dynamic portfolio problems with a risk-free asset and risky asset and proposed a fuzzy control model. Sadjadi etc. [5], who researched the fuzzy portfolio selection problem with different borrowing and lending rate. Huang and Qiao [6] tried to study the multi period portfolio problem under fuzzy environment, and proposed a risk index of an uncertain multi period portfolio problem. Liu [7] proposed a pair of two-level mathematical programs to calculate the upper bound and lower bound of return and transfer it into a pair of ordinary one-level linear programs.

This paper is organized as follows. In the next section, we present the fuzzy number and its four arithmetic operates. In section 3, we introduce the formulation of transformation for a fuzzy portfolio model to a crisp programming. Section 4 presents a numerical example with real data from the Chinese stock market. The paper ends with some concluding remarks.

2 Notation of fuzzy numbers

From [8], membership function of a trapezoidal fuzzy number $\tilde{A}=(r_1,r_2,r_3,r_4)$ is defined as

$$\mu_{\tilde{A}}(x)=\begin{cases}0, x<r_1\\ \dfrac{x-r_1}{r_2-r_1}, r_1\le x<r_2\\ 1, r_2\le x<r_3\\ \dfrac{r_4-x}{r_4-r_3}, r_3\le x<r_4\\ 0, x\ge r_4\end{cases}$$

, where r_2,r_3 are the left and right modal values and r_1,r_4 are the left and right spreads.

Triangular fuzzy number is a special case of trapezoidal fuzzy number, for any $M=(r_1,r_2,r_3,r_4)$, which is a trapezoidal fuzzy number, when $r_2=r_3$, trapezoidal fuzzy number will degenerate to the triangular fuzzy number $M=(r_1,r_2,r_3)$.

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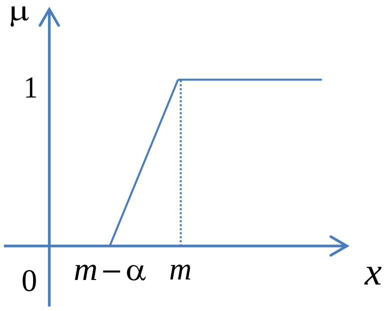

FIGURE 1

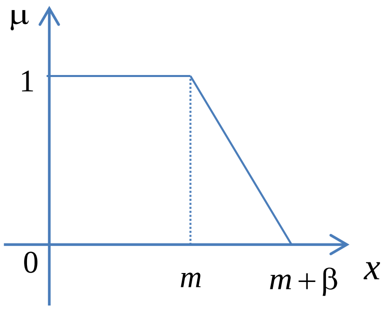

FIGURE 2

There exists certain flexibility in making the portfolio decision .Fig 1 denotes the degree of satisfaction of the expected value of the portfolio. If the expected value is larger than $m-\alpha$, the degree of satisfaction increase, when the expected value is larger than m the investor is completely satisfy. Fig 2 shows the degree of satisfaction of the budget spending. If the budget spending is less than m, the investor is completely satisfy. However, if the budget spending is greater than m, then the degree of satisfaction decrease. When the budget spending is greater than $m+\beta$, the degree of satisfaction becomes zero.

The four arithmetic operates on a trapezoidal fuzzy number as follows:

Suppose $M_1=(r_1,r_2,r_3,r_4)$, $M_2=(s_1,s_2,s_3,s_4)$ are two trapezoidal fuzzy numbers, then $M_1+M_2=(r_1+s_1,r_2+s_2,r_3+s_3,r_4+s_4)a$ for any $\lambda\in R$, $\lambda M_1=\begin{cases}(\lambda r_1,\lambda r_2,\lambda r_3,\lambda r_4),\lambda\geq 0\\(\lambda r_4,\lambda r_3,\lambda r_2,\lambda r_1),\lambda<0\end{cases}$, $M_1\times M_2=(r_1s_1,r_2s_2,r_3s_3,r_4s_4),r_1,s_1>0$.

3 Problem statement and modelling

3.1 INTRODUCTION TO QUALITATIVE POSSIBILITY THEORY

This section introduces qualitative possibility theory, see to Dubois and Prade [9], which is used to deal with the fuzzy constraint involving both uncertain and flexible parameters. The basic concept is introduced as follows.

Let U be a set of states and X be a set of possible consequences, a possibility distribution π represent the incomplete knowledge on the state on U and X be the possibility distribution representing the preference of decision maker on X. The utility of a decision d whose consequence in state u is $x=d(u)$ for $u\in U$ can be evaluated by combining the plausibilities $\pi(u)$ and the utilities $u(x)$ in a suitable way. Two quantitative criteria were proposed by Dubois and Prade to evaluate the worth of decision d regarding uncertain information:

Pessimistic criterion:

$$U_*(d)=\inf_{u\in U}\max\left(1-\pi(u),\mu\left(d(u)\right)\right). \tag{1}$$

Optimistic criterion:

$$U^*(d)=\sup_{u\in U}\min\left(\pi(u),\mu\left(d(u)\right)\right). \tag{2}$$

In this paper, the pessimistic criterion is used to determine the satisfaction degree of the fuzzy constraint that contains uncertain parameters on its left-hand side and flexible parameters on its right-hand side. For example, the left-hand side is the future fuzzy cost, and the right-hand side is the flexible budget of a investor. The satisfaction degree of the decision d can be defined

$$C_d\left(\tilde{R},\tilde{B}\right)=\inf\max\left(1-\mu_{\tilde{R}}(x),\mu_{\tilde{B}}(x)\right), \tag{3}$$

where $\tilde{R}$ is the possible consequences of decision d and $\tilde{B}$ is the preference of decision maker about the consequence.

3.2 MODEL FORMULATION OF THE PORTFOLIO SELECTION

The multi-period portfolio selection problem is to select a set of strategy to maximize the expected benefits during the planning horizon under some budget constraints. Since imprecision and flexibility are encountered in making portfolio decisions, a fuzzy programming model is proposed here to optimize portfolio decisions in an uncertain environment.

Notation

n the total number of candidate assets

$\tilde{B}_t$ the flexible budget available for stage t

r_{it} fuzzy future value of candidate asset i at stage t

x_{it} share of asset i at stage t

b_{it} share of buying asset i at stage t, $c_{it}(b_{it})$ is buying cost

s_{it} share of selling asset i at stage t, $c_{it}(s_{it})$ is selling cost

I_{pt} the return of portfolio at stage t

W_0 the initial wealth

I_t the given return of a investor

I_t' degree of tolerance for deviation, equivalents to risk

The optimal model: max $\prod_{t=1}^{T}(1+I_{pt})W_0$, where $I_{pt}=\sum_{i=1}^{n}\left[r_{it}(x_{it}+b_{it}-s_{it})-c_{it}(b_{it})-c_{it}(s_{it})\right]$,

$$s.t\quad\begin{cases}|I_{pt}-I_t|\le I_t'\\ \sum_{i=1}^{n}\left[c_{it}(b_{it})+c_{it}(s_{it})\right]\le\tilde{B}_t\\ x_{it},b_{it},s_{it}\ge 0\end{cases}.$$

In fact, $x_{it}=x_{i,t-1}+b_{it}-s_{it}$. And for general, For simple, given the original respective share are $\frac{1}{n}$, the cost of buying and selling asset is proportional, i.e., for example, transaction costs function is $C(b_{it})=0.008b_{it}$ and $C(s_{it})=0.008s_{it}$, $I_t=0.1, I_t'=0.02$. The purpose is to maximize wealth, and the first constraint means risk, the second constraint means budget.

3.3 QUALITATIVE POSSIBILITY THEORY TO TRANSFORMATION

In this paper, the approach of Inuiguchi and Ramik [10] has been extended with qualitative possibility theory to handle the fuzzy constraint containing both uncertain and flexible parameters. Then the fuzzy portfolio selection model is transformed into a linear programming model which can be solved by an optimization technique.

Consider an inequality constraint (first constraint) of the above problem, it can be divided to two inequalities: $I_{pt}\le 0.12, I_{pt}\ge 0.08$. For $I_{pt}\le 0.12$, it equals to $\sum_{i=1}^{n}\left[r_{it}(x_{it}+b_{it}-s_{it})-0.008b_{it}-0.008s_{it}\right]\le 0.12$.

Set $x_{it}'=x_{it}+b_{it}-s_{it}$, $b_{it}'=\sum_{i=1}^{n}(0.008b_{it}+0.008s_{it})+0.12$, then the above constraint is transferred to $\sum_{i=1}^{n}r_{it}x_{it}'\le b_{it}'$.

If the decision maker feels that the satisfaction degree of the constraint needs to be greater than or equal to λ_i, the constraint can be reformulated based on pessimistic criterion according to Eq. (3): $C\left(r_{1t}x_{1t}'+r_{2t}x_{2t}'+\cdots r_{nt}x_{nt}', b_{it}'\right)\ge\lambda_i$

Then the constraint can be transformed into $\sum_{i=1}^{n}r_{it}^{c2}x_{it}'+\lambda_i\sum_{i=1}^{n}r_{it}^{r}x_{it}'\le b_{it}'$, where r_{it}^{c2} is the right modal values of r_{it}, and r_{it}^{r} is their right spreads, respectively. Similarly, we can transform another constraints $I_{pt}\ge 0.08$ and $\sum_{i=1}^{n}\left[c_{it}(b_{it})+c_{it}(s_{it})\right]\le\tilde{B}_t$ into a crisp objective function. So far, all the constraints are transformed to linear programming.

Next for the fuzzy objective function. Suppose that it is satisfied for an investor when the satisfaction degree of the objective function should be greater than or equal to γ.

max υ, $s.t.\ C(W_T,\upsilon)\ge\gamma$, where $W_T=\prod_{t=1}^{T}(1+I_{pt})W_0$.

In order to avoid multiplication, we logarithmic the target function $\ln(W)_T=\sum_{t=1}^{T}\ln\left(1+\sum_{i=1}^{n}\left(r_{it}x_{it}'-0.008(b_{it}+s_{it})\right)\right)+\ln(W_0)$.

So target of maxing the return equals to max $\sum_{t=1}^{T}\ln\left(1+\sum_{i=1}^{n}\left(r_{it}x_{it}'-0.008(b_{it}+s_{it})\right)\right)$.

Let $C^{c1}=\sum_{t=1}^{T}\ln\left(1+\sum_{i=1}^{n}\left(r_{it}^{c1}x_{it}'-0.008(b_{it}+s_{it})\right)\right)$ and $C^{l}=\sum_{t=1}^{T}\ln\left(1+\sum_{i=1}^{n}\left(r_{it}^{l}x_{it}'-0.008(b_{it}+s_{it})\right)\right)$.

The target function is equivalent to max υ, $s.t\ \ C^{c1}-\gamma C^{l}\ge\upsilon$.

Since the problem is the maximization problem, the left edge of $C(W_T,\upsilon)$ is used based on pessimistic criterion to determine the satisfaction degree of the portfolio value that is greater than γ.

4 Illustrated example

This section presents an example of portfolio selection problem to illustrate the approach developed. The investment has three stages. The preferred budgets for stages 1, 2, and 3 are described in fuzzy numbers (in millions): (0, 271.2, 0, 40), (0, 984.9, 0, 200), and (0, 1975.8, 250), respectively. That is $B_1=(0,271.2,0,40)$, $B_2=(0,984.9,0,200)$, $B_3=(0,1975.8,0,250)$.

Let the target satisfaction degree of objective function (γ) be set to 0.95 and risk, budget (λ) are set to 0.9 for all t and i, respectively. Table 1 lists the uncertain return for three stages in fuzzy numbers.

TABLE 1 Fuzzy return for 3 assets

Asset no	Fuzzy return		
	Stage 1	*Stage 2*	*Stage 3*
1	(0.3,0,3,0.045,0.045)	(0.5,0,5,0.075,0.075)	(0.45,0,45,0.067,0.067)
2	(0.1,0,1,0.015,0.015)	(0.35,0,35,0.011,0.011)	(0.2,0,2,0.015,0.015)
3	(0.1,0.1,0.0 15, 0.0 15)	(0.75,0.75, 0.112, 0.112)	(0.5, 0.5, 0.015, 0.015)
4	(0.55, 0.55, 0.075, 0.075)	(0.65,0.65,0.0975,0.097)	(0.17,0.17,0.025,0.025)
5	(0.2, 0.2, 0.03, 0.03)	(0.85, 0.85, 0.012, 0.012)	(0.2, 0.2, 0.03, 0.03)

From solving the model, the dynamic portfolio is:

At stage 1, $b_{11}=0.7203, a_{21}=a_{31}=0, a_{41}=0.314, a_{51}=0$ and $s_{11}=s_{21}=s_{31}=0$, $s_{41}=s_{51}=0$.

At stage 2, $a_{12}=0, a_{22}=0.577, a_{32}=0, a_{42}=0.027, a_{52}=0$ and $s_{21}=s_{22}=s_{32}=0$, $s_{42}=0, s_{52}=0.516$.

At stage 3, $a_{51}=0.183, a_{52}=0.204, a_{53}=a_{54}=a_{55}=0$ and $s_{51}=0, s_{52}=0$, $s_{53}=0.1265, s_{54}=s_{35}=0$.

5 Conclusion

This paper developed a fuzzy multi- portfolio selection model to determine invest strategy that maximizes the target portfolio value while there is lack of reliable information. The fuzzy portfolio selection model developed was able to handle both uncertain and flexible parameters and the proposed possibilistic transformation method can convert the model into a crisp mathematical model, which can be solved by linear programming.

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