The continuous-time optimal portfolio using a multivariate normal inverse Gaussian model

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Abstract

This paper develops the continuous-time portfolio model using a multivariate normal inverse Gaussian model. Though the weighted average of lognormal variables is no longer lognormal, it can be approximated by other distributions, such as a multivariate normal inverse Gaussian model. Our method belongs to the analytic approximation class. By comparing to Monte Carlo experiments, it illustrates the computational efficiency and accuracy of our approach.

Keywords: Continuous-time portfolio, Normal inverse Gaussian, Approximation, Monte Carlo, Optimization

1 Introduction

The corns torn of portfolio selection problem is stem from Markowitz (1952) [1] on mean-variance model for single period portfolio selection problem. After Markowitz's pioneer work, numerous scholars extended the single period case to multi-period ones, and continuous-time framework. Bielecki et al (2005) [2] considered bankruptcy prohibition in continuous time with martingale approach. Czichowsky and Schweizer (2011) [3] proposed cone- constrained continuous-time mean-variance portfolio problem with price processes being semi-martingales. Some literates aim to the market condition under continuous-time environment. Li et.al (2002) [4] supposed the price processes of assets are continuous Ito process, and derived the optimal portfolio for the continuous-time mean-variance model with no shorting using duality method. Fu (2010) [5] derived explicit closed form solutions for the dynamic mean-variance portfolio selection problem with borrowing constraint, the method used is the HJB equation of stochastic programming. Cui (2014) [6] considered the mean-variance formulation in multi-period portfolio selection under no shorting constraint.

To the best our knowledge, few of all the existing research focus on the price process of weighted sum of assets, in fact, the continuous-time portfolio payoff depends on the value of a portfolio of assets. The challenge in describing the portfolio stem from the fact that there is no explicit closed form for the weighted sum of correlated assets. There are two categories approximation techniques to solve this problem, numerical methods and approximations. Although numerical methods such as Monte Carlo simulation is a very flexible method, it is very time-consuming. Jarrow and Rudd (1982) [7] is the first to introduce Edeworth expansion. Turnbull and wakeman (1991) [8] used an Edeworth series expansion to approximation the density function of the weighted sum. Mileusky (1998) [9] adopted the reciprocal Gramma distribution for alternative. Because a normal inverse Gaussian process incorporates an idiosyncratic drift, characteristics volatility, correlated Brownian motion, and a common inverse Gaussian time change, the multivariate normal inverse Gaussian model should provide more realistic diffusion of assets. This paper adopts a multi-normal inverse Gaussian (MNIG) process approximation to the weighted sum of correlated assets.

The plan for the paper is as follows. It is details the MNIG process in section 2. In section 3, we propose the approximation method. Section 4, introduces the optimal portfolio selection model and give the compared results of our method to Monte Carlo experiment in a numerical example. Section 5 contains our conclusion.

2 A multi-normal inverse Gaussian distribution

In the following, we will introduce the notions of ING and MING process according to Wu (2009) [10]. Suppose that G follows an inverse Gaussian distribution with parameter $a,b>0$, whose density function is as follows

$$f\left(x\right)=\sqrt{\frac{1}{2\pi x^3}}\exp\left(-\frac{b\left(x-a\right)}{2a^2x}\right),x\geq 0.$$

And its characteristic function is given by

$$\varphi\left(u\right)=\exp\left(\frac{b}{a}\left(1-\sqrt{1-\frac{2a^2ui}{b}}\right)\right).$$

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A direct calculation yields the mean and variant are a and $\frac{a^3}{b}$, respectively. Due to the stochastic process is dependent on time t. It needs to introduce G_t whose parameters are at and bt. So $E(G_t)=at$, $Var(G_t)=\frac{a^3t^2}{b}$. For simply, let $a=1, b=\frac{1}{\gamma}$.

We define a normal inverse Gaussian process $X^j(t)=\theta_j G_t + W^j(G_t)$, where W_{Gt} is Brownian motion [11], θ determines the tendency of the sample paths. The mean and variant of $X^j(t)$ are θ_j and $\sigma_j^2+\gamma\theta_j^2$, σ_j is the volatility rate, and the characteristic function is $exp\left(\gamma^{-1}\left(1-\sqrt{1+u^2\sigma_j^2\gamma-2u\theta_j\gamma i}\right)\right)$.

3 Model formulations

It is considered a portfolio consisting of n assets with price $S_t^i=S_0^i\, exp\left((r-q_i)t+X_t^j+d_j t\right)$, where $d_j=exp\left(\gamma^{-1}\left(\sqrt{1-\sigma_j^2\gamma-2\theta_j\gamma j}-1\right)\right)$.

The log-value of portfolio is $s_t=ln S_t=\sum_{i=1}^{n}\omega_i\, ln S_t^i$.

The characteristic function of s_T is as $\varphi_{s_T}(u)=exp\left(iu(aT-qT+s_0)T+k^{-1}T\left(1-\sqrt{1+u^2\sigma^2k-2i\theta uk}\right)\right)$

$$d=\sum_{i=1}^{n}\omega_i d_i, \theta=\sum_{i=1}^{n}\omega_i\theta_i,$$
$$q=\sum_{i=1}^{n}\omega_i q_i, S_0=\sum_{i=1}^{n}\omega_i S_0^i, k=\gamma T^{-1},$$
$$\sigma=\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\omega_i\omega_j\rho_{ij}\sigma_i\sigma_j},$$
$$w=exp\left(k^{-1}\left(\sqrt{1-\sigma^2k-2\theta k}-1\right)\right).$$
$$a=r+d+w$$

The detailed derivation refer to Wu. The relationship of the characteristic function and moment [12] is as follow proposition.

Proposition. Suppose that there exist n order moments of variable X, then the characteristic function of X also exist, and its n order derivative exit, $E(X^k)=\frac{\phi^k{}_X(0)}{i^k}$.

In fact, $\phi_X^k(t)=\int_{-\infty}^{+\infty} i^k exp(itx) f(x)dx=i^k E\left(X^k exp(itX)\right)$.

Let $t=0$, $E(X^k)=\frac{\phi_X^k(0)}{i^k}$.

So $E(X)=\frac{\phi_X'(0)}{i}$, $E(X^2)=-\phi_X''(0)$.

From the characteristic function of s_T, we get $Es_T=aT-qT+T\theta+S_0$, $Es_T^2=(aT-qT+\theta T+S_0)^2+T\sigma^2$.

4 Numerical example

In this section, we firstly describe the optimal portfolio model, then give an numerical example to illustrate the accuracy and computational efficiency of our method.

4.1 THE OPTIMAL PORTFOLIO MODEL

We consider the relatively simple continuous-time mean-variance portfolio selection model refers to the problem of finding the optimal admission strategy to minimize the variance while attaining a given level of the expectation

$$\begin{cases} \min\limits_{\omega} Var(s_T)=E(s_T-u)^2 \\ s.t\ E(s_T)=u \end{cases}, \quad (1)$$

where u is a given constant, representing the expected level, which the investor requires to achieve, $E(s_T), Var(s_T)=E(s_T^2)-(E(s_T))^2$ is from section 3.

According to the optimal portfolio model (1), the problem can be deal with by the Lagrange method. Introducing the Lagrange multiplier λ leads to the following problem

$$min\, E(X(T)-u)^2+2\lambda(EX(T)-u). \quad (2)$$

Let $\pi(T)=g(x(T),\lambda)$ be the optimal solution of the Lagrangian problem (2) and $G(x_0,\lambda)$ be the optimal value. According to the Lagrange duality theory, if λ^* satisfies $\max\limits_{\lambda} G(x_0,\lambda)$, then $\pi^*(t)=g(x(t),\lambda^*)$ is the optimal shares of (1) and $G(x_0,\lambda^*)$ is its optimal value. Problem (2) is equivalent to $min\, E(x(T)+a)^2$, where $a=\lambda-u$.

4.2 ILLUSTRATIVE EXAMPLE

The goal of our numerical experiments is to test t the computational efficiency and accuracy of our approach. We therefore set up a simulation study and compare the results to the calculated results using Monte Carlo simulations.

First, we define a two-asset portfolio with a one year, a constant continuously compounded risk-free rate of 0.1. Each asset is given its own dynamic parameters: the drift parameters θ_1 and θ_2 are 0.1 and 0.2, and the volatility parameters σ_1 and σ_2 are 0.2 and 0.3. We also consider two different levels of correlation and weights between the underlying assets: the correlation ρ is set to either 0.5 or 0, and the weights are set to either (0.7, 0.3) or (0.3, 0.7) under two different economic states corresponding to $\gamma = 0.1$ and 0.2. To solve problem (1), the Monte Carlo simulation results of portfolio weights is $\omega_1 = 0.2813, \omega_2 = 0.7187$, and the results from the approximation method proposed in our study is $\omega_1 = 0.2752, \omega_2 = 0.7248$.

From the compared results, we find that the portfolio shares from our model and Monte Carlo simulation is nearly the same.

5 Conclusion

The difficult to solve the continuous-time portfolio selection is a closed-form formula of the weights of assets is not available. The main contribution of this paper is to develop a closed-form analytic expression for the portfolio. Our approach is based on a multivariate normal inverse Gaussian (mNIG) model, which is a more appropriate representation of asset dynamics than the geometric Brownian motion (GBM) model. Because an NIG model has economic meaning: θ and σ represent the drift and volatility of the individual assets respectively, while γ represents the effect of an economic state shared by all assets. Numerical example results for two-asset shows the accuracy and computational efficiency compared to Monte Carlo simulation results. It provides a new way to deal with the continuous-time portfolio selection.

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