Simulations of standard Brownian motion

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Abstract

This paper investigates algorithms for simulation of the trajectories of a Brownian motion (Wiener process) with given accuracy and reliability. Spectral representation of Wiener process as random series examines as a model. Estimates of the accuracy and reliability investigated in various function spaces - spaces of measurable integrated functions, Orlicz spaces and spaces of continuous functions. Given the accuracy of the numbers and simulation algorithms error of Gaussian random variables in the model are used strictly sub-Gaussian random variables. Examples of simulation are represented below.

Keywords: Wiener process, simulation, sub-Gaussian model, accuracy and reliability

1 Introduction

Statistical models of Wiener processes used to solve many practical problems, particularly, in the calculation of integrals over Wiener processes or in the numerical solution of stochastic differential equations [1, 2].

As a model, we consider the spectral decomposition processes as a random series or integrals.

Moment of difference of model and process is an estimation of model accuracy in the problems of statistical simulation [1, 2]. In papers [3, 4] investigated accuracy and reliability estimation of the simulation of random processes in various functional spaces.

In this paper, we investigate the accuracy and reliability of the Wiener process simulation in various functional spaces - $L_2, L_p, C,$ Orlicz spaces.

2 Basic definitions

Let (T, Ξ, μ) - be some measurable space and $\mu(T) = 1$.

Definition 1. Generalized Wiener process with parameter $\alpha, \alpha \in (0,1]$ will be called a Gaussian random process with zero mean and correlation function

$$R_\alpha(t,s) = EW_\alpha(t)W_\alpha(s) = \frac{1}{2}\left(|t|^{2\alpha} + |s|^{2\alpha} - |t-s|^{2\alpha}\right).$$

When $\alpha = \frac{1}{2}$ we have a classical Wiener process. This paper considers the classical Wiener process. As a model of the Wiener process we consider random series

$$S_M(t) = \sum_{i=1}^{M} f_i(t)\eta_i ,$$

where $\{\eta_i\}$ - Gaussian random variables from $N(0,1)$ (in the general case is optionally dependent).

Let all $S_M(t)$ and $W_\alpha(t)$ belong to some function space $A(T)$.

Definition 2. Model $S_M(t)$ approximates the process $W_\alpha(t)$ with specified accuracy $\delta > 0$ and reliability $0 < \varepsilon < 1$ in the norm of a function space $A(T)$, in case

$$P\left\{\left\|W_\alpha(t) - S_M(t)\right\|_A > \delta\right\} \le 1 - \varepsilon .$$

When simulating a sequence of Gaussian random variables due to the accuracy of computational tools, simulation algorithms obtain sub-Gaussian random variables.

Definition 3. Random variable ζ called sub-Gaussian, if for $\forall\lambda \quad \exists a$ takes place

$$E\exp(\lambda\zeta) \le \exp\left(\frac{a^2\lambda^2}{2}\right).$$

Space of sub-Gaussian random variables is a Banach space with norm

$$\tau(\zeta) = \inf\left\{a \ge 0 : E\exp\{\lambda\zeta\} \le \exp\left\{\frac{a^2\lambda^2}{2}\right\}, \lambda \in R^1\right\}.$$

When $E\zeta^2 = a^2$ we have a strictly sub-Gaussian random variables.

Definition 4. C is a continuous, steam, convex function $U(x)$, such as $U(0) = 0$, $U(x) > 0$ when $x \ne 0$.

Definition 5. Orlicz space generated by the C- function $U(x)$ is a family of functions $\{f(t), \quad t \in T\}$ such as, for every $f(t)$ exists a constant r such as $\int_T U\left(\frac{f(t)}{r}\right) d\mu(t) < \infty$.

Orlicz space is a Banach space under the norm

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$$\|f\|_{L_U} = \inf\left\{ r > 0 : \int_T U\left(\frac{f(t)}{r}\right) d\mu(t) \le 1 \right\}.$$

Will consider C-function $U(x)$ for which function $G_U(t) = \exp\left\{\left(U^{(-1)}(t-1)\right)^2\right\}$ is convex and when $t \ge 1$, where $U^{(-1)}(x)$ is inverse function of $U(x)$.

This condition is fulfilled for C-function

$$U(x) = \exp\left\{|x|^a\right\} - 1, \quad 1 \le a \le 2,$$

then $G_U(t) = \exp\left\{(\ln t)^{\frac{2}{a}}\right\}$.

Theorem [5]. For any real $T \ne 0$ and any real t_0 each of random functions $\sqrt{T}W\left(\frac{t}{T}\right)$, $W(t+t_0) - W(t_0)$ and $|t|W\left(\frac{1}{t}\right)$ is similar to random function $W(t)$.

Thus, without loss of generality, we can consider the Wiener process on the interval $T = [0,1]$ and consider different views of the Wiener process in the form of series.

Decomposition of the Wiener process in the Eigen functions of the correlation operator of the Brownian bridge $t \in [0,1]$ has the form [6]

$$\xi_1(t) = t\eta_0 + \sqrt{2}\sum_{i=1}^{\infty} \frac{\sin(i\pi t)}{i\pi}\eta_i,$$

where $\{\eta_0, \eta_1, \eta_2, \ldots\}$ - independent standard Gaussian random variables.

$\lambda_i = i\pi$ - the eigenvalues of the correlation operator. Fourier series expansion on $t \in [0,1]$ [5]

$$\xi_2(t) = t\eta_0 + \sqrt{2}\sum_{i=1}^{\infty}\left(\eta_{1i}\frac{\sin(2\pi it)}{2\pi i} + \eta_{2i}\frac{1-\cos(2\pi it)}{2\pi i}\right),$$

where $\{\eta_{1i}, \eta_{2i}\}$ - independent standard Gaussian random variables.

3 Estimation of Karhunen – Loeve model

As a model for the expansion of the Wiener process in the eigenfunctions of the correlation operator consider

$$S_1(t,M) = t\eta_0 + \sqrt{2}\sum_{i=1}^{M} \frac{\sin(i\pi t)}{i\pi}\eta_i.$$

Based on the results [3, 4] it is easy to obtain the following assertion.

Assertion 1. Model $S_1(t,M)$ approximates the process $\xi_1(t)$ with accuracy $\delta > 0$ and reliability $1-\varepsilon$, $0 < \varepsilon < 1$:

a) in $L_2([0,1])$ if inequalities hold $\delta^2 > J1_{M+1}$

$$\exp\left\{\frac{1}{2}\right\}\frac{\delta}{\sqrt{J1_{M+1}}}\exp\left\{-\frac{\delta^2}{2J1_{M+1}}\right\} \le \varepsilon$$

or $\delta^2 > J1_{M+1}$.

$$\left(\frac{\delta^2 - J1_{M+1}}{J2_{M+1}} + 1\right)^{\frac{1}{2}} \exp\left\{-\frac{\delta^2 - J1_{M+1}}{2J2_{M+1}}\right\} \le \varepsilon,$$

where

$$J1_{M+1} = \sum_{i=M+1}^{\infty} \lambda_i^{-2} \text{ and } J2_{M+1} = \left(\sum_{i=M+1}^{\infty} \lambda_i^{-4}\right)^{\frac{1}{2}}.$$

b) in $L_p([0,1])$, $p > 1$, $p \ne 2$ or inequalities hold

$$\exp\left\{\frac{1}{2}\right\}\frac{\delta}{\sigma_{M+1}}\exp\left\{-\frac{\delta^2}{2\sigma_{M+1}^2}\right\} \le \varepsilon,$$

where $\sigma_{M+1}^2 = \sup_{t\in[0,1]}\left(\sum_{i=M+1}^{\infty} \frac{\sin^2(i\pi t)}{(i\pi)^2}\right)$

and inequalities $\delta^2 > \sigma_{M+1}^2$ when $1 \le p < 2$ or $\delta^2 > (p+1)\sigma_{M+1}^2$ when $p > 2$.

c) in $L_U([0,1])$ if inequalities hold

$$\exp\left\{\frac{1}{2}\right\}\frac{\delta U^{(-1)}(1)}{\sigma_{M+1}}\exp\left\{-\frac{\delta^2\left(U^{(-1)}(1)\right)^2}{2\sigma_{M+1}^2}\right\} \le \varepsilon,$$

$$\delta^2 > \left(2 + \left(U^{(-1)}(1)\right)^{-2}\right)\sigma_{M+1}^2.$$

d) in $C([0,1])$ if for $\beta \in \left(0, \frac{1}{2}\right]$ inequalities hold.

$$2\exp\left\{-\frac{1}{2}\left(\frac{\delta}{G_{M+1}}\right)^2+1+\sqrt{2}\left(\frac{\delta}{G_{M+1}}\right)^{\frac{4\beta+1}{2\beta+1}}\left(F_\beta+\frac{\pi}{2}\right)\right\}\times\exp\left\{2\left(\frac{\delta}{G_{M+1}}\right)^{\frac{4\beta}{2\beta+1}}\left(F_\beta q_\beta\left(\frac{\delta}{G_{M+1}}\right)+\frac{\pi^2}{8}\left(\frac{\delta}{G_{M+1}}\right)^{\frac{1-2\beta}{1+2\beta}}\right)\right\}\le\varepsilon$$

and $\delta > 2G_{M+1}$, where $G_{M+1}=\left(\sum_{i=M+1}^{\infty}\lambda_i^{-2}\right)^{\frac{1}{2}}$,

$$F_\beta=\sum_{i=M+1}^{\infty}\left|\ln\left(w_R^{(-1)}\left(\frac{1-\left(1-2G_{M+1}^2\delta^{-2}\right)^{\frac{1}{2}}}{\pi i}\right)\right)\right|^{\frac{1}{2}}\frac{\pi^2 i^2}{G_i^{2(1-\beta)}},\ w_R(h)=\sup_{|u-v|\le h}\left(\int_0^1\left(R(u,x)-R(v,x)\right)^2dx\right)^{\frac{1}{2}},$$

$$q_\beta=\begin{cases}1, & \beta\in\left[\frac{1}{6},\frac{1}{2}\right],\\ \left(\frac{\delta}{G_{M+1}}\right)^{\frac{1-6\beta}{2(2\beta+1)}}, & \beta\in\left(0,\frac{1}{6}\right).\end{cases}$$

Since $\sup_{t\in[0,1]}\left|\sin(i\pi t)\right|\le 1$, then $\sigma_{M+1}^2=\sum_{i=M+1}^{\infty}\lambda_i^{-2}$.

For implementations simulation and given δ and ε let's find M. The Table 1 shows the estimates for M in the various functional spaces. Calculations for $L_U([0,1])$ is not represented. Depending on the function $U(x)$ results are between $L_2([0,1])$ and $C([0,1])$.

Figure 1 shows the implementation of a Wiener process to represent $\xi_1(t)$.

TABLE 1 Values of M for different functional spaces

δ	ε	a) $L_2([0,1])$	b) $L_2([0,1])$	$C([0,1])$
0.1	0.05	110	36	10000
0.05	0.05	750	120	100000
0.01	0.05	18700	2250	>1000000
0.1	0.01	260	45	
0.05	0.01	1050	130	
0.01	0.01	26000	2350	

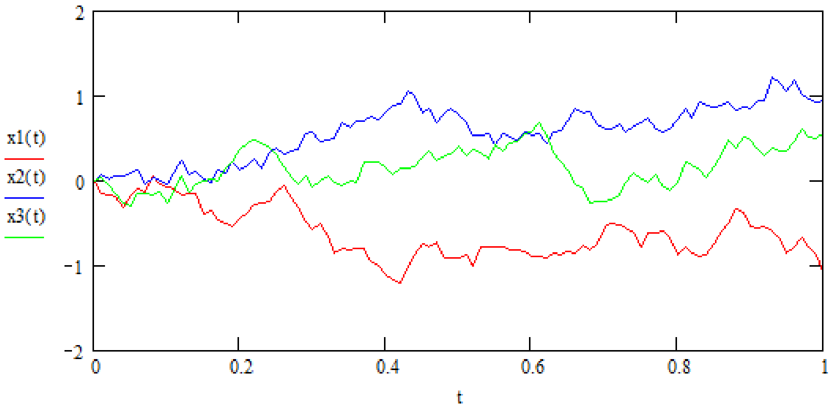

FIGURE 1 Implementation of the Weiner process

4 Estimation of Fourier model

As a model for the expansion of the Wiener process in the Fourier series consider

$$S_2(t,M)=t\eta_0+\sqrt{2}\sum_{i=1}^{M}\left(\eta_{1i}\frac{\sin(2\pi it)}{2\pi i}+\eta_{2i}\frac{1-\cos(2\pi it)}{2\pi i}\right).$$

Based on the results [3, 4] it is easy to obtain the following assertion.

Assertion 2. Model $S_2(t,M)$ approximates process $\xi_2(t)$ with accuracy $\delta>0$ and reliability $1-\varepsilon$, $0<\varepsilon<1$:

a) in $L_2([0,1])$ if inequalities hold $\delta^2 > J1_{M+1}$

$$\exp\left\{\frac{1}{2}\right\}\frac{\delta}{\sqrt{J1_{M+1}}}\exp\left\{-\frac{\delta^2}{2J1_{M+1}}\right\}\le\varepsilon$$

or $\delta^2 > J1_{M+1}$

$$\left(\frac{\delta^2 - J1_{M+1}}{J2_{M+1}}+1\right)^{\frac{1}{2}}\exp\left\{-\frac{\delta^2 - J1_{M+1}}{2J2_{M+1}}\right\}\le\varepsilon$$

where

$$J1_{M+1} = 2\sum_{i=M+1}^{\infty}(\pi i)^{-2} \text{ and } J2_{M+1} = 2\left(\sum_{i=M+1}^{\infty}(\pi i)^{-4}\right)^{\frac{1}{2}}.$$

b) in $L_p([0,1])$, $p>1$, $p\ne 2$ if inequalities hold

$$\exp\left\{\frac{1}{2}\right\}\frac{\delta}{\sigma_{M+1}}\exp\left\{-\frac{\delta^2}{2\sigma_{M+1}^2}\right\}\le\varepsilon$$

where

$$\sigma_{M+1}^2 = 2\sup_{t\in[0,1]}\left(\sum_{i=M+1}^{\infty}\left(\left(\frac{\sin(2\pi it)}{2\pi i}\right)^2+\left(\frac{1-\cos(2\pi it)}{2\pi i}\right)^2\right)\right)$$

and inequalities $\delta^2 > \sigma_{M+1}^2$ when $1\le p<2$

or $\delta^2 > (p+1)\sigma_{M+1}^2$ when $p>2$.

c) in $L_U([0,1])$ if inequalities hold

$$\exp\left\{\frac{1}{2}\right\}\frac{\delta U^{(-1)}(1)}{\sigma_{M+1}}\exp\left\{-\frac{\delta^2\left(U^{(-1)}(1)\right)^2}{2\sigma_{M+1}^2}\right\}\le\varepsilon,$$

$$\delta^2 > \left(2+\left(U^{(-1)}(1)\right)^{-2}\right)\sigma_{M+1}^2.$$

d) in $C([0,1])$ in case of $\beta\in\left(0,\frac{1}{2}\right]$ inequalities hold

$$2\exp\left\{-\frac{1}{2}\left(\frac{\delta}{2G_{M+1}}\right)^2+1+\sqrt{2}\left(\frac{\delta}{2G_{M+1}}\right)^{\frac{4\beta+1}{2\beta+1}}\left(F_\beta+\frac{\pi}{2}\right)\right\}\times\exp\left\{2\left(\frac{\delta}{2G_{M+1}}\right)^{\frac{4\beta}{2\beta+1}}\left(F_\beta q_\beta\left(\frac{\delta}{2G_{M+1}}\right)+\frac{\pi^2}{8}\left(\frac{\delta}{2G_{M+1}}\right)^{\frac{1-2\beta}{1+2\beta}}\right)\right\}\le\varepsilon$$

and $\delta > 4G_{M+1}$, where $G_{M+1} = \left(\sum_{i=M+1}^{\infty}(2\pi i)^{-2}\right)^{\frac{1}{2}}$

$$F_\beta = 4\pi^2\sum_{i=M+1}^{\infty}\left|\ln\left(w_R^{(-1)}\left(\frac{1-\left(1-8G_{M+1}^2\delta^{-2}\right)^{\frac{1}{2}}}{\pi i}\right)\right)\right|^{\frac{1}{2}}\frac{i^2}{G_i^{2(1-\beta)}},$$

$$q_\beta = \begin{cases} 1, & \beta\in\left[\frac{1}{6},\frac{1}{2}\right], \\ \left(\frac{\delta}{2G_{M+1}}\right)^{\frac{1-6\beta}{2(2\beta+1)}}, & \beta\in\left(0,\frac{1}{6}\right). \end{cases}$$

The Table 2 shows the estimates for the M in various functional spaces for presentation $\xi_2(t)$. Figure 2 shows the implementation of the Wiener process for submission $\xi_2(t)$.

TABLE 2 Values of M for different functional spaces

δ	ε	a) $L_2([0,1])$	b) $L_2([0,1])$	$C([0,1])$
0.1	0.05	380	60	300000
0.05	0.05	1500	220	1000000
0.01	0.05	38000	4400	>1000000
0.1	0.01	400	70	
0.05	0.01	2100	240	
0.01	0.01	58000	4500	

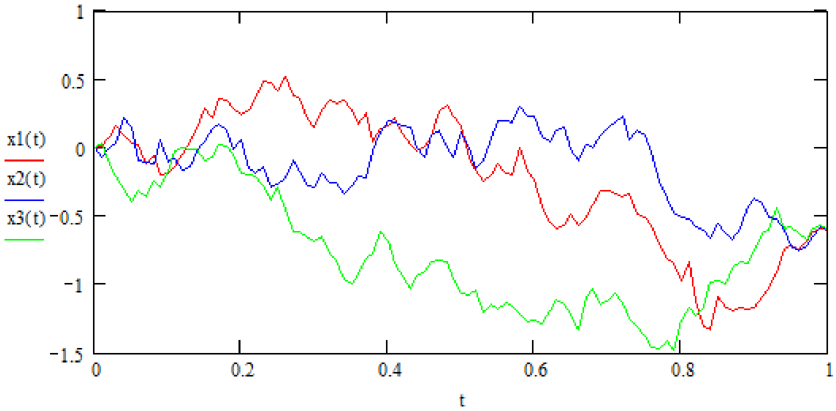

FIGURE 2 Implementation of Wiener process

When constructing a model we use strictly sub-Gaussian random variables obtained by the Equation $\eta_i = \sum_{j=1}^{12} \gamma_j - 6$, where $\{\gamma_j\}$ - uniformly distributed on $[0,1]$ random numbers. When building multiple implementations - the algorithm has a natural parallelization. Figure 3 shows the implementation of $\xi_1(t)$ and $\xi_2(t)$ retrieved from one sequence of strictly sub-Gaussian random variables. Figure 4 shown the implementation of presentation $\xi_1(t)$ at various M ($M = 110$ and $M = 36$).

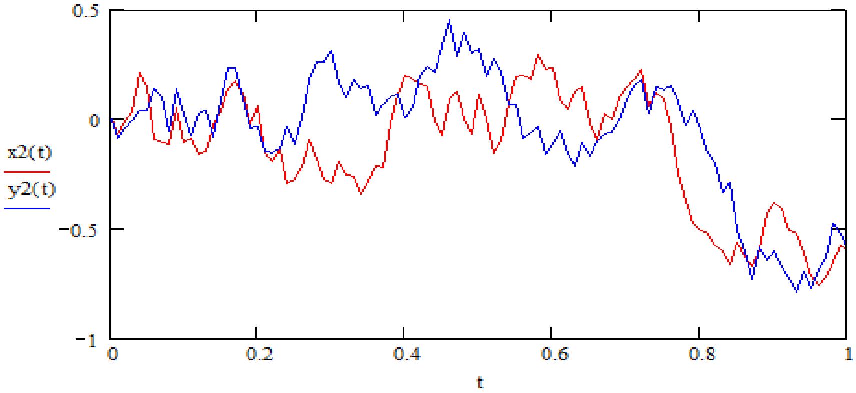

FIGURE 3 Implementation of the Wiener process for the views $\xi_1(t)$ and $\xi_2(t)$

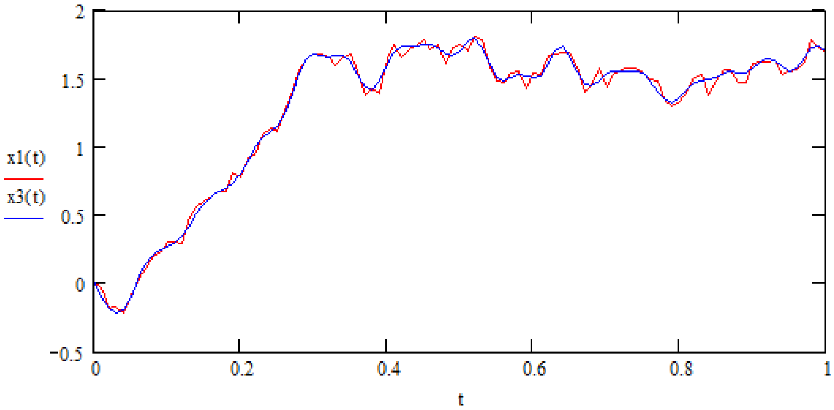

FIGURE 4 An example: Implementation of the Wiener process for the view $\xi_1(t)$ when *M=110* and *M=36*

5 Conclusions

We obtain estimates for the construction of strictly sub-Gaussian model of Wiener process. The model is constructed with the specified accuracy and reliability for various functional spaces. We found the implementation of the Wiener process for different views. Has interest for obtaining similar estimates for the generalized Wiener process.

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