Duality for multi-objective semi-infinite programming with $K-(F_b,\rho)-$ convexity

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Abstract

In this paper, some nonsmooth generalized convex functions called uniform $K-(F_b,\rho)-$ convex function, uniform $K-(F_b,\rho)-$ pseudoconvex function, uniform $K-(F_b,\rho)-$ quasiconvex function are defined using $K-$ directional derivative and $K-$ subdifferential. Nonsmooth multi-objective semi-infinite programming involving these generalized convex functions is researched, some Mond-Weir type duality results are obtained.

Keywords: Nonsmooth, Multi-objective Semi-infinite Programming, Mond-Weir type duality, Uniform $K-(F_b,\rho)-$ Convex Function

1 Introduction

The convexity theory plays an important role in many aspects in mathematical programming. In recent years, to relax convexity assumption involved in sufficient conditions for optimality or duality theorems, various generalizations of convex functions have appeared in the literature. Hanson and Mond introduced type I and type II function [1]. Reuda and Hanson extended type I function and obtained pseudo type I and quasi type I function [2]. Bector and Singh introduced $b-$ convex function [3]. Bector, Suneja and Cupta extended $b-$ convex function and defined univex function [4]. Mishra discussed the optimality and duality for multi-objective programming with generalized univexity [5]. Preda introduced $(F_b,\rho)-$ convex function as extension of $F-$ convex function and $\rho-$convex function [6, 7, 8].

In this paper, we introduce a new classes of generalized convex functions, that is, uniform $K-(F_b,\rho)-$ convex function, uniform $K-(F_b,\rho)-$ pseudoconvex function, uniform $K-(F_b,\rho)-$ quasiconvex functioned. Then we consider nonsmooth multi-objective semi-infinite programming involving these generalized convex functions and obtain some Mond-Weir type duality results.

2 Definitions

Throughout this paper, let R^n be the $n-$dimensional Euclidean space and R^n be its non-negative orthat. Now we consider the following multi-objective semi-infinite programming problem:

$$(VP)\begin{cases}\min f(x)=(f_1(x),f_2(x),\cdots,f_p(x))\\ s.t.\, g(x,u)\le 0, x\in X, u\in U\end{cases}$$

, where X is an open subset of R^n, $f:X\to R^p$, $g:X\times U\to R^n$, $U\subset R$ is an infinite parameter set.

Let $A=\{x\mid g(x,u)\le 0, x\in X, u\in U\}$, $\Delta=\{i\mid g(x,u^i)\le 0, x\in X, u^i\in U\}$, $I(x^*)=\{i\mid g(x^*,u^i)\le 0, x^*\in X, u^i\in U\}$, $U^*=\{u^i\in U\mid g(x,u^i)\le 0, x\in X, i\in\Delta\}$ is any countable subset of U, $\Lambda=\{\mu_j\mid \mu_j\ge 0|, j\in\Delta,$ there is only finite μ_j such that $\mu_j\ne 0\}$.

Notations. If $x,y\in R^n$, then $x\le y\Leftrightarrow x_i\le y_i$, $i=1,2,\cdots,n$, and there exists at least one $i_0\in\{1,2,\cdots,n\}$ such that $x_{i_0}<y_{i_0}$.

Definition 2.1 [9] Let $K(\cdot,\cdot)$ is a local cone approximation, the function $f^K(x,\cdot):X\to R$ with $f^K(x;y):=\inf\{\xi\in R\mid (y,\xi)\in K(epif,(x,f(x)),y\in R^n)\}$ is called $K-$directional derivative of f at x.

Definition 2.2 [9] A function $f:X\to R$ is called $K-$ subdifferentiable at x if there exists a convex compact set $\partial^K f(x)$ such that $f^K(x,y)=\max_{\xi\in\partial^K f(x)}<\xi,y>,\forall y\in R^n$, where $\partial^K f(x):=\{x^*\in X^*\mid <y,x^*>\le f^K(x;y),\forall y\in R^n\}$ is called $K-$subdifferential of f at x.

Definition 2.3 A functional $F:X\times X\times R^n\to R$ ($X\subset R^n$) is called sublinear with respect to the third variable, if for any $x_1,x_2\in X$,

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(i) $F(x_1,x_2;a_1+a_2)\le F(x_1,x_2;a_1)+F(x_1,x_2;a_2),\forall a_1,a_2\in R^n$;

(ii) $F(x_1,x_2;ra)=rF(x_1,x_2;a),\forall r\in R, r\ge 0, a\in R^n$.

Definition 2.4 $x^*\subset X^0$ is called an efficient solution for (VP) if and only if there exists no $x\subset X^0$ such that $f(x^*)\le f(x)$.

In the following definitions, we suppose $C\subset R^n$ is a nonempty set, $x_0\in C$, $f:C\to R$ is a local Lipschitz function at x_0, $F:C\times C\times R^n\to R$ is sublinear with respect to the third variable, $\phi:R\to R$, $b:C\times C\times[0,1]\to R_+$, $\lim\limits_{\lambda\to 0^+}b(x,x_0;\lambda)=b(x,x_0)$, $d(\cdot,\cdot)$ is a pseudo-metric in R^n. In [9], Elster and Thierfelder defined $K-$directional derivative and $K-$subdifferential and pointed out that $K-$subdifferential is most generalized. Now we will define some new generalized convex functions using $K-$ directional derivative and $K-$subdifferential.

Definition 2.5 A function $f:C\to R$ is said to be uniform $K-(F_b,\rho)-$ convex at x_0 with respect to F,ϕ,b,d, if for all $x\in C$, there exists $\rho\in R$ such that $b(x,x_0)\phi[f(x)-f(x_0)]\ge F(x,x_0;\xi)+\rho d^2(x,y),\forall\xi\in\partial^K f(x_0)$.

Definition 2.6 A function $f:C\to R$ is said to be strictly uniform $K-(F_b,\rho)-$ convex at x_0 with respect to F,ϕ,b,d, if for all $x\in C, x\ne x_0$, there exists $\rho\in R$ such that $b(x,x_0)\phi[f(x)-f(x_0)]>F(x,x_0;\xi)+\rho d^2(x,x_0),\forall\xi\in\partial^K f(x_0)$.

Definition 2.7 A function $f:C\to R$ is said to be uniform $K-(F_b,\rho)-$ pseudoconvex at x_0 with respect to F,ϕ,b,d, if for all $x\in C$, there exists $\rho\in R$ such that $b(x,x_0)\phi[f(x)-f(x_0)]<0\Rightarrow F(x,x_0;\xi)+\rho d^2(x,y)<0,\forall\xi\in\partial^K f(x_0)$.

Definition 2.8 A function $f:C\to R$ is said to be strictly uniform $K-(F_b,\rho)-$ pseudoconvex at x_0 with respect to F,ϕ,b,d, if for all $x\in C, x\ne x_0$, there exists $\rho\in R$ such that $b(x,x_0)\phi[f(x)-f(x_0)]\le 0\Rightarrow F(x,x_0;\xi)+\rho d^2(x,y)<0,\forall\xi\in\partial^K f(x_0)$.

Definition 2.9 A function $f:C\to R$ is said to be uniform $K-(F_b,\rho)-$quasiconvex at x_0 with respect to F,ϕ,b,d, if for all $x\in C$, there exists $\rho\in R$ such that $b(x,x_0)\phi[f(x)-f(x_0)]\le 0\Rightarrow F(x,x_0;\xi)+\rho d^2(x,y)\le 0,\forall\xi\in\partial^K f(x_0)$.

Definition 2.10 A function $f:C\to R$ is said to be weak uniform $K-(F_b,\rho)-$quasiconvex at x_0 with respect to F,ϕ,b,d, if for all $x\in C$, there exists $\rho\in R$ such that $b(x,x_0)\phi[f(x)-f(x_0)]<0\Rightarrow F(x,x_0;\xi)+\rho d^2(x,y)\le 0,\forall\xi\in\partial^K f(x_0)$.

3 Mond-Weir type duality

We consider the following Mond-Weir type dual programming for (VP):

$$(VD)\begin{cases}\max f(y)\\ s.t. 0\in\partial^K(\sum_{i=1}^p\lambda_i f_i)(y)+\sum_{j\in\Delta}\mu_j\partial^K g(y,u^j),\\ \sum_{j\in\Delta}\mu_j g(y,u^j)\ge 0,\\ \lambda_i\ge 0, i=1,2,\cdots,p,\sum_{i=1}^p\lambda_i=1,\\ \mu_j\in\Lambda.\end{cases}$$

Let $W=\Big\{(y,u^j,\lambda,\mu)\,|\,0\in\partial^K(\sum_{i=1}^p\lambda_i f_i)(y)+\sum_{j\in\Delta}\mu_j\partial^K g(y,u^j),\sum_{j\in\Delta}\mu_j g(y,u^j)\ge 0,\ \lambda_i\ge 0,\ i=1,2,\cdots,p,\sum_{i=1}^p\lambda_i=1,\mu_j\in\Lambda,u^j\in U^*,U^*\subset U\Big\}$.

Theorem 3.1 (Weak duality) Assume that $x\in A$, $(y,u^j,\lambda,\mu)\in W$, if for any $\lambda_i>0, i=1,2,\cdots,p$, $\mu_j\in\Lambda, j\in\Delta$, there exist $F,\phi_1,\phi_2,b_1,b_2,\rho_1\in R^1$, $\rho_2^j\in R^{|I|}$ such that

(i) $\sum_{i=1}^p\lambda_i f_i$ is uniform $K-(F_b,\rho_1)-$ convex or strictly uniform $K-(F_b,\rho_1)-$convex at y;

(ii) $-b_2(x,y)\phi_2[\sum_{j\in I(y)}\mu_j g(y,u^j)]\ge F(x,y;\sum_{j\in I(y)}\mu_j\eta_j)+\sum_{j\in I(y)}\mu_j\rho_2^j d^2(x,y)$, $\forall\eta_j\in\partial^K g(y,u^j),u^j\in U^*,j\in I(y)$;

(iii) $\alpha<0\Rightarrow\phi_1(\alpha)<0,\alpha\ge 0\Rightarrow\phi_2(\alpha)\ge 0$, $b_1(x,y)>0,b_2(x,y)\ge 0$;

(iv) $\rho_1+\sum_{j\in\Delta}\mu_j\rho_2^j\ge 0$.

Then the following inequality cannot hold: $f(x)\le f(y)$.

Proof

Suppose that $f(x)\le f(y)$, then there exists i_0 such that $f_{i_0}(x)<f_{i_0}(y),\quad f_i(x)<f_i(y),\forall i\ne i_0$.

Since $\lambda_i>0,i=1,2,\cdots,p$, we have $\sum_{i=1}^p\lambda_i f_i(x)<\sum_{i=1}^p\lambda_i f_i(y)$.

By hypothesis (iii), we have

$b_1(x,y)\phi_1[\sum_{i=1}^{p}\lambda_i f_i(x)-\sum_{i=1}^{p}\lambda_i f_i(y)]<0.$

By hypothesis (i), we get

$$F(x,y;\xi)+\rho_1 d^2(x,y)<0,\forall\xi\in\partial^K(\sum_{i=1}^{p}\lambda_i f_i)(y), \qquad (1)$$

observe that $\mu_j\in\Lambda$ and $g(y,u^j)=0, j\in I(y)$, we have $\sum_{j\in I(y)}\mu_j g(y,u^j)\geq 0.$

By hypothesis (iii), we have $-b_2(x,y)\phi_2[\sum_{j\in I(y)}\mu_j g(y,u^j)]\leq 0.$

By hypothesis (ii), we get $F(x,y;\sum_{j\in I(y)}\mu_j\eta_j)+\sum_{j\in I(y)}\mu_j\rho_2^j d^2(x,y)\leq 0,$ $\forall\eta_j\in\partial^K g(y,u^j),u^j\in U^*, j\in I(y)$.

Let $\mu_j=0,$ $j\notin I(y),$ we have $F(x,y;\sum_{j\in\Delta}\mu_j\eta_j)+\sum_{j\in\Delta}\mu_j\rho_2^j d^2(x,y)\leq 0,$

$$\forall\eta_j\in\partial^K g(y,u^j),u^j\in U^*, j\in\Delta. \qquad (2)$$

adding (1) and (2) ,using the sublinearity of F, we can obtain $F(x,y;\xi+\sum_{j\in\Delta}\mu_j\eta_j)+(\rho_1+\sum_{j\in\Delta}\mu_j\rho_2^j)d^2(x,y)<0.$

By hypothesis (iv), we have $F(x,y;\xi+\sum_{j\in\Delta}\mu_j\eta_j)<0.$

So $\xi+\sum_{j\in\Delta}\mu_j\eta_j\neq 0$, which contradicts the first subjective condition of (VD).

Theorem 3.2 (Weak duality) Assume that $x\in A,$ $(y,u^j,\lambda,\mu)\in W,$ if for any $\lambda_i>0, i=1,2,\cdots,p,$ $\mu_j\in\Lambda, j\in\Delta,$ there exist $F,\phi_1,\phi_2,b_1,b_2,\rho_1\in R^1,$ $\rho_2^j\in R^{|I|}$ such that

(i) $\sum_{i=1}^{p}\lambda_i f_i$ is uniform $K-(F_b,\rho_1)-$ pseudoconvex at y;

(ii) $-b_2(x,y)\phi_2[\sum_{j\in I(y)}\mu_j g(y,u^j)]\leq 0\Rightarrow$ $F(x,y;\sum_{j\in I(y)}\mu_j\eta_j)+\sum_{j\in I(y)}\mu_j\rho_2^j d^2(x,y)\leq 0,$ $\forall\eta_j\in\partial^K g(y,u^j),u^j\in U^*, j\in I(y)$;

(iii) $\alpha<0\Rightarrow\phi_1(\alpha)<0,\alpha\geq 0\Rightarrow\phi_2(\alpha)\geq 0,$ $b_1(x,y)>0,b_2(x,y)\geq 0$;

(iv) $\rho_1+\sum_{j\in\Delta}\mu_j\rho_2^j\geq 0.$

Then the following inequality cannot hold: $f(x)\leq f(y).$

Theorem 3.3 (Weak duality) Assume that $x\in A,$ $(y,u^j,\lambda,\mu)\in W,$ if for any $\lambda_i\geq 0,$ $i=1,2,\cdots,p,$ $\mu_j\in\Lambda, j\in\Delta,$ there exist $F,\phi_1,\phi_2,b_1,b_2,\rho_1\in R^1,$ $\rho_2^j\in R^{|I|}$ such that

(i) $\sum_{i=1}^{p}\lambda_i f_i$ is strictly uniform $K-(F_b,\rho_1)-$ pseudo-convex at y;

(ii) $-b_2(x,y)\phi_2[\sum_{j\in I(y)}\mu_j g(y,u^j)]\leq 0\Rightarrow$ $F(x,y;\sum_{j\in I(y)}\mu_j\eta_j)+\sum_{j\in I(y)}\mu_j\rho_2^j d^2(x,y)\leq 0,$ $\forall\eta_j\in\partial^K g(y,u^j),u^j\in U^*, j\in I(y)$;

(iii) $\alpha\leq 0\Rightarrow\phi_1(\alpha)\leq 0,\alpha\geq 0\Rightarrow\phi_2(\alpha)\geq 0,$ $b_1(x,y)\geq 0,b_2(x,y)\geq 0$;

(iv) $\rho_1+\sum_{j\in\Delta}\mu_j\rho_2^j\geq 0.$

Then the following inequality cannot hold: $f(x)\leq f(y).$

Theorem 3.4 (Weak duality) Assume that $x\in A,$ $(y,u^j,\lambda,\mu)\in W,$ if for any $\lambda_i\geq 0,$ $i=1,2,\cdots,p,$ $\mu_j\in\Lambda, j\in\Delta,$ there exist $F,\phi_1,\phi_2,b_1,b_2,\rho_1\in R^1,$ $\rho_2^j\in R^{|I|}$ such that

(i) $\sum_{i=1}^{p}\lambda_i f_i$ is uniform $K-(F_b,\rho_1)-$ quasiconvex at y;

(ii) $-b_2(x,y)\phi_2[\sum_{j\in I(y)}\mu_j g(y,u^j)]\leq 0\Rightarrow$ $F(x,y;\sum_{j\in I(y)}\mu_j\eta_j)+\sum_{j\in I(y)}\mu_j\rho_2^j d^2(x,y)<0,$ $\forall\eta_j\in\partial^K g(y,u^j),u^j\in U^*, j\in I(y)$;

(iii) $\alpha\leq 0\Rightarrow\phi_1(\alpha)\leq 0,\alpha\geq 0\Rightarrow\phi_2(\alpha)\geq 0,$ $b_1(x,y)>0,b_2(x,y)\geq 0$;

(iv) $\rho_1+\sum_{j\in\Delta}\mu_j\rho_2^j\geq 0.$

Then the following inequality cannot hold: $f(x)\leq f(y).$

Theorem 3.5 (Weak duality) Assume that $x\in A,$ $(y,u^j,\lambda,\mu)\in W,$ if for any $\lambda_i>0,$ $i=1,2,\cdots,p,$ $\mu_j\in\Lambda, j\in\Delta,$ there exist $F,\phi_1,\phi_2,b_1,b_2,\rho_1\in R^1,$ $\rho_2^j\in R^{|I|}$ such that

(i) $\sum_{i=1}^{p}\lambda_i f_i$ is weak uniform $K-(F_b,\rho_1)-$ quasiconvex at y;

(ii) $-b_2(x,y)\phi_2[\sum_{j\in I(y)}\mu_j g(y,u^j)]\leq 0\Rightarrow$ $F(x,y;\sum_{j\in I(y)}\mu_j\eta_j)+\sum_{j\in I(y)}\mu_j\rho_2^j d^2(x,y)<0,$ $\forall\eta_j\in\partial^K g(y,u^j),u^j\in U^*, j\in I(y)$;

(iii) $\alpha<0\Rightarrow\phi_1(\alpha)<0,\alpha\geq 0\Rightarrow\phi_2(\alpha)\geq 0,$

$b_1(x,y)>0, b_2(x,y)\ge 0$;

(iv) $\rho_1+\sum_{j\in\Delta}\mu_j\rho_2^j \ge 0$.

Then the following inequality cannot hold: $f(x)\le f(y)$.

The proofs of Theorem 3.2 —Theorem 3.5 are similar to Theorem 3.1.

Theorem 3.6 (Strong duality) Assume that x^* is an efficient solution for (VP), if for any $\lambda_i>0,\ i=1,2,\cdots,p$, $\mu_j\in\Lambda,\ j\in\Delta$, there exist $F,\phi_1,\phi_2,b_1,b_2,\rho_1\in R^1$, $\rho_2^j\in R^{|I|}$ such that

(i) $\sum_{i=1}^{p}\lambda_i f_i$ is uniform $K-(F_b,\rho_1)-$ pseudoconvex at y;

(ii) $-b_2(x,y)\phi_2[\sum_{j\in I(y)}\mu_j g(y,u^j)]\le 0\Rightarrow$

$F(x,y;\sum_{j\in I(y)}\mu_j\eta_j)+\sum_{j\in I(y)}\mu_j\rho_2^j d^2(x,y)\le 0,$

$\forall\eta_j\in\partial^K g(y,u^j), u^j\in U^*, j\in I(y)$;

(iii) $\alpha<0\Rightarrow\phi_1(\alpha)<0, \alpha\ge 0\Rightarrow\phi_2(\alpha)\ge 0,$

$b_1(x,y)>0, b_2(x,y)\ge 0$;

(iv) $\rho_1+\sum_{j\in\Delta}\mu_j\rho_2^j\ge 0.$;

(v) The *Kuhn-Tuker* condition is hold at x^*, that is, there exist $\lambda^*\in R_+^p, \mu^*\in R_+^{|I|}$ such that

$$0\in\partial^K(\sum_{i=1}^{p}\lambda_i^* f_i)(x^*)+\sum_{j\in I(x^*)}\mu_j^*\partial^K g(x^*,u^j),$$

$$\mu_j^* g(x^*,u^j)=0,\ j\in\Delta,$$

$$(\lambda^*,\mu^*)\ge 0, \sum_{i=1}^{p}\lambda_i^*=1.$$

Then there exist (λ^*,μ^*) such that $(x^*,u^j,\lambda^*,\mu^*)$ is an efficient solution for (VD), and the optimal values of (VP) and (VD) are equal.

Proof

Since x^* is an efficient solution for problem (VP) and the *Kuhn-Tuker* condition is hold at x^*, therefore, for any $u^j\in U^*$, there exist $\lambda_i^*>0,\ i=1,2,\cdots,p,$ $\sum_{i=1}^{p}\lambda_i^*=1$, $\mu_j^*\in\Lambda, j\in\Delta$ such that

$$0\in\partial^K(\sum_{i=1}^{p}\lambda_i^* f_i)(x^*)+\sum_{j\in\Delta}\mu_j^*\partial^K g(x^*,u^j),$$

$$\sum_{j\in\Delta}\mu_j^* g(x^*,u^j)\ge 0.$$

So $(x^*,u^j,\lambda^*,\mu^*)$ is a feasible solution for (VD).

Suppose that $(x^*,u^j,\lambda^*,\mu^*)$ is not an efficient solution for (VD), then there exist (y,u^j,λ,μ) and i_0 such that $f_{i_0}(x^*)<f_{i_0}(y),\ f_i(x^*)\le f_i(y),\forall i\ne i_0$. that is,

$$f(x^*)\le f(y), \tag{3}$$

which contradicts the conclusion of Theorem 3.2. Therefore, $(x^*,u^j,\lambda^*,\mu^*)$ is an efficient solution for (VD). Obviously, the optimal values of (VP) and (VD) are equal.

4 Conclusions

In this paper, we introduce a new classes of generalized convex functions, that is, uniform $K-(F_b,\rho)-$ convex function, uniform $K-(F_b,\rho)-$ pseudoconvex function, uniform $K-(F_b,\rho)-$ quasiconvex function, etc. Then we consider nonsmooth multi-objective semi-infinite programming involving these generalized convex functions and proved some weak and strong duality theorems of Mond-Weir type duality.

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