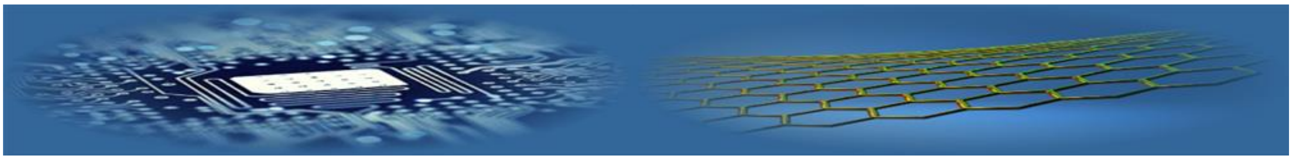

Relationships among convergence concepts of uncertain sequences

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Abstract

Uncertain sequence is a sequence of uncertain variables indexed by integers. In this paper, a new kind of sequence convergence that complete convergence was presented. Then, the relationships among complete convergence, convergence in p-distance, convergence in measure, convergence in distribution, convergence uniformly almost surely and convergence almost surely were investigated.

Keywords:

uncertain measure
uncertain variable
expectation
convergence

1 Introduction

In order to describe subjective uncertain phenomenon, Liu [6] founded uncertainty theory which based on uncertain measure that satisfies normality, duality, subadditivity, and product axiom in 2007, then refined by Liu [11] in 2010. Next, some properties of uncertain measure were studied by Gao [3]. Thereafter, Liu [6] introduced uncertain variable which is a measurable function from an uncertainty space to the set of real numbers. A sufficient and necessary condition of uncertainty distribution was proved by Peng and Imamura [16] in 2010. After introduced the definition of independence by Liu [9], Liu [11] presented the operational law of uncertain variables. Up to now, uncertainty theory has already applied to uncertain programming (Liu [8]), uncertain risk analysis (liu [13]), uncertain process (Liu [7], Liu [19]), and uncertain logic (Liu [14], Li and Liu [10]), etc. In addition, uncertain calculus which deals with differentiation and integration of uncertain processes was given by Liu [9]. Then uncertain differential equation was founded by Liu [7] and further researched by Yao and Chen [20] whose main contents include a concept of α -path to uncertain differential equation and a numerical method is designned for solving uncertain differential equations. Gao [4] and Chen [1] both researched uncertain inference which is a process of deriving consequences from human knowledge via uncertain set theory (Liu [12], Liu [15]).

Sequence convergence plays an important role in the fundamental theory of mathematics, therefore Liu [7] introduced some concepts of sequence convergence in uncertainty theory and discussed their relationships. Then another type of convergence named convergence uniformly almost surely was presented by You [18]. Xia [17] investigated the convergence of uncertain sequences which contains Cauchy convergence of uncertain sequences and the sufficient conditions of convergence almost surely. Thereafter, the dual convergence of uncertain sequence was investigated by Yuan, Zhu and Guo [21]. Guo, Zhu and Yuan [5] presented a necessary and sufficient condition of convergence in measure for uncertain sequences. In addition, the convergence concepts of complex uncertain sequence were proposed by Chen, Ning and Xiao [2] in 2016.

In this paper, we will give a new concept of convergence of uncertain sequences and discuss the relationships among the basic definitions of convergence. The rest of this paper is organized as follows. Uncertainty space and some basic contents and theorems of uncertainty theory will be introduced in Section 2. Then the relationships among complete convergence, convergence in p-distance, convergence in measure, convergence in distribution, convergence uniformly almost surely and convergence almost surely will be investigated in Section 3. At last, a brief summary is given.

2 Preliminary

In this section, we presented some definitions and theorems in uncertain environment which will be used in this paper.

Let Γ be a nonempty set, and let L be a σ −algebra over Γ . A number $M\{\Lambda\}$ indicates the level that each element $\Lambda \in L$ (which is called an event) will occur. Liu [6] proposed the set function M, which is called uncertain measures if it satisfies the following three axioms:

Axiom 1 (Normality) $M\{\Gamma\}=1$.

Axiom 2 (Self-Duality) $M\{\Lambda\}+M\{\Lambda^c\}=1$, for any event Λ.

Axiom 3 (Subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have

$$M\{\bigcup_{i=1}^{\infty}\Lambda_i\} \le \sum_{i=1}^{\infty} M\{\Lambda_i\}. \quad (1)$$

Next, the definition of uncertain space is introduced.

Definition 2.1 (Liu [6]) Let Γ be a nonempty set, let L be a σ − algebra over Γ, and let M be an uncertain measure. Then the triple (Γ, L, M) is called an uncertainty space.

The properties of uncertain measure is recalled in the following theorem.

Theorem 2.1 (Monotonicity, Liu [6]) Uncertain measure M is a monotone increasing set function. That is, for any events $\Lambda_1 \subset \Lambda_2$, we have $M\{\Lambda_1\} \le M\{\Lambda_2\}$.

Definition 2.2 (Liu [6]) An uncertain variable is a function ξ from an uncertainty space (Γ, L, M) to the set of real numbers such that $\{\xi \in B\}$ is an event for any set B of real numbers.

Definition 2.3 (Liu [6]) Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} M\{\xi \ge x\}dx - \int_{-\infty}^0 M\{\xi \le x\}dx, \quad (2)$$

provided that at least one of the two integrals is finite.

Theorem 2.2 (Liu [6]) Let ξ be an uncertain variable with uncertainty distribution Φ. Then

$$E[\xi] = \int_0^{+\infty} (1-\Phi(x))dx - \int_{-\infty}^0 \Phi(x)dx. \quad (3)$$

Theorem 2.3 (Markov Inequality, Liu [6]) Let ξ be an uncertain variable. Then for any give numbers $t > 0$ and $p > 0$, we have

$$M\{|\xi| \ge t\} \le \frac{E[|\xi|^p]}{t^p}. \quad (4)$$

Then, we reviewed some convergence concepts of uncertain sequence and some theorems about them.

Definition 2.4 (Liu [6]) The uncertain sequence $\{\xi_i\}$ is said to be convergent a.s. to ξ if there exists an event Λ with $M\{\Lambda\} = 1$ such that

$$\lim_{i\to\infty} |\xi_i(\gamma) - \xi(\gamma)| = 0, \quad (5)$$

for event $\gamma \in \Lambda$. In that case we write $\xi_i \to \xi$, a.s.

Theorem 2.4 (You [18]) Let $\xi, \xi_1, \xi_2, \cdots$ be uncertain variables. Then $\{\xi_i\}$ convergent a.s. to ξ if and only if for any $\varepsilon > 0$, we have

$$M\{ \bigcap_{m=1}^{\infty}\bigcup_{i=m}^{\infty}(\gamma \in \Gamma \| \xi_i(\gamma) - \xi(\gamma)| \ge \varepsilon) \} = 0. \quad (6)$$

Definition 2.5 (Liu [6]) The uncertain sequence $\{\xi_i\}$ is said to be convergent in measure to ξ if

$$\lim_{i\to\infty} M\{|\xi_i - \xi| \ge \varepsilon\} = 0 \quad (7)$$

for every $\varepsilon > 0$.

Definition 2.6 (Liu [6]) Let $\Phi, \Phi_1, \Phi_2, \cdots$ be the uncertainty distributions of uncertain variables $\xi, \xi_1, \xi_2, \cdots$, respectively. We say the uncertain sequence $\{\xi_i\}$ converges in distribution to ξ if

$$\lim_{i\to\infty} \Phi_i(x) = \Phi(x) \quad (8)$$

for all x at which $\Phi(x)$ is continuous.

Definition 2.7 (You [18]) The uncertain sequence $\{\xi_i\}$ is said to be convergent uniformly a.s. to ξ if and only if

$$\lim_{i\to\infty} M\{\bigcup_{i=m}^{\infty}(\gamma \in \Gamma \| \xi_i(\gamma) - \xi(\gamma)| \ge \varepsilon)\} = 0. \quad (9)$$

Definition 2.8 (You and Yan [22]) The uncertain sequence $\{\xi_i\}$ is said to be convergent in p-distance to ξ if

$$\lim_{i\to\infty} d_p(\xi_i, \xi) = \lim_{i\to\infty}(E[|\xi_i - \xi|^p])^{\frac{1}{p+1}} = 0. \quad (10)$$

Theorem 2.5 (Liu [6]) If the uncertain sequence $\{\xi_i\}$ converges in measure to ξ, then $\{\xi_i\}$ converges in distribution to ξ.

Theorem 2.6 (You [18]) Suppose $\xi, \xi_1, \xi_2, \cdots$ are uncertain variables. If $\{\xi_i\}$ converges uniformly a.s. to ξ, then $\{\xi_i\}$ converges in measure to ξ.

3 Relationships among convergence concepts

A new definition of convergence of uncertain sequence is given in this section. Then, we discuss the relationships among these convergence concepts mentioned in Section 2.

Now we show the relationships among convergence in p-distance, convergence in measure, and convergence in distribution.

Theorem 3.1 Suppose that $\xi, \xi_1, \xi_2, \cdots$ are uncertain variables defined on uncertainty space (Γ, L, M). If $\{\xi_i\}$ converges in p-distance to ξ, then $\{\xi_i\}$ converges in measure to ξ.

Proof: If uncertain sequence $\{\xi_i\}$ converges in p-distance to ξ, then we have $\lim_{i\to\infty} E[|\xi_i - \xi|^p] = 0$. It follows from Theorem 2.3 that for any given number $\varepsilon > 0$, we have

$$M\{|\xi_i - \xi| \ge \varepsilon\} \le \frac{E[|\xi_i - \xi|^p]}{\varepsilon^p} \to 0, \quad (11)$$

as $i \to \infty$. Thus uncertain sequence $\{\xi_i\}$ converges in measure to ξ. The theorem is proved.

Example 3.1 Convergence in measure does not imply convergence in p-distance. For example, take an uncertainty space (Γ, L, M) to be $\{\gamma_1, \gamma_2, \cdots\}$ with $M\{\gamma_i\} = \frac{1}{2i}$. The uncertain variables are defined by

$$\xi_i(\gamma_j) = \begin{cases} 2i, & if \quad i = j \\ 0, & otherwise \end{cases}, \quad (12)$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. For some small number $\varepsilon > 0$ and $i > 1$, we have

$$M\{|\xi_i - \xi| \ge \varepsilon\} = M\{\gamma \| \xi_i(\gamma) - \xi(\gamma)| \ge \varepsilon\} \\ = M\{\gamma_i\} = \frac{1}{2i} \to 0 , \tag{13}$$

as $i \to \infty$. Thus, the sequence $\{\xi_i\}$ converges in measure to ξ.

However, for each $i > 1$, we have the uncertainty distribution of uncertain variable $|\xi_i - \xi|$ is

$$\Phi_i(x) = \begin{cases} 0, & if\ x < 0 \\ 1 - \frac{1}{2i}, & if\ 0 \le x < 2i \\ 1, & if\ x \ge 2i. \end{cases} \tag{14}$$

Then according to Theorem 2.2, for each $i > 1$, we have

$$E[|\xi_i - \xi|] = \int_{2i}^{\infty} (1-1)dx + \int_0^{2i} (1 - (1 - \frac{1}{2i}))dx = 1 \tag{15}$$

It follows from Theorem 2.1 that

$$E[|\xi_i - \xi|^p] = \int_0^{\infty} M\{|\xi_i - \xi|^p \ge x\}dx \\ \ge \int_0^{\infty} M\{|\xi_i - \xi| \ge x\}dx \\ = E[|\xi_i - \xi|] = 1 . \tag{16}$$

Therefore,

$$\lim_{i\to\infty} d_p(\xi_i, \xi) = \lim_{i\to\infty} (E[|\xi_i - \xi|^p])^{\frac{1}{p+1}} = 1 \ne 0 \tag{17}$$

That is, the uncertain sequence $\{\xi_i\}$ does not converges in p-distance to ξ.

Theorem 3.2 Let $\xi, \xi_1, \xi_2, \cdots$ be uncertain variables defined on uncertainty space (Γ, L, M). If uncertain sequence $\{\xi_i\}$ converges in p-distance to ξ, then $\{\xi_i\}$ converges in distribution to ξ.

Proof: By Theorem 2.5, we know that the uncertain sequence $\{\xi_i\}$ convergence in measure means it convergence in distribution. Then, it follows from Theorem 3.1 that $\{\xi_i\}$ converges in distribution to ξ.

Example 3.2 Convergence in distribution does not imply convergence in p-distance. Take an uncertainty space (Γ, L, M) to be $\{\gamma_1, \gamma_2\}$ with $M\{\gamma_1\} = M\{\gamma_2\} = \frac{1}{2}$.

The uncertain variable are defined by

$$\xi(\gamma) = \begin{cases} -a, & if\ \gamma = \gamma_1 \\ a, & if\ \gamma = \gamma_2 \end{cases}, \tag{18}$$

where a is a positive number. We also define $\xi_i = -\xi$ for $i = 1, 2, \cdots$. Thus ξ_i and ξ have the same uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & if\ x < -a \\ \frac{1}{2}, & if\ -a \le x \le a \\ 1, & if\ x > a. \end{cases} \tag{19}$$

That is to say, uncertain sequence $\{\xi_i\}$ converges in distribution to ξ. Then we have $|\xi_i - \xi|_p = (2a)^p$, for $\gamma = \gamma_1, \gamma_2$ and its expected value is

$$E[|\xi_i - \xi|_p] = \int_0^{(2a)^p} 1dx = (2a)^p . \tag{20}$$

Therefore,

$$\lim_{i\to\infty} d_p(\xi_i, \xi) = \lim_{i\to\infty} (E[|\xi_i - \xi|^p])^{\frac{1}{p+1}} = (2a)^{\frac{p}{p+1}} . \tag{21}$$

Then $\{\xi_i\}$ does not converge in p-distance to ξ.

Next, we give a new convergence concept of uncertain sequence which is complete convergence.

Definition 3.1 Let $\xi, \xi_1, \xi_2, \cdots$ be uncertain variables defined on uncertainty space (Γ, L, M). Then $\{\xi_i\}$ is said to be completely convergent to ξ if

$$\lim_{i\to\infty} \sum_{k=i}^{\infty} M\{|\xi_k - \xi| \ge \varepsilon\} = 0 , \tag{22}$$

for any $\varepsilon > 0$.

Theorem 3.3 Suppose $\xi, \xi_1, \xi_2, \cdots$ are uncertain variables defined on uncertainty space (Γ, L, M). If $\{\xi_i\}$ completely converges to ξ, then $\{\xi_i\}$ converges uniformly almost surely to ξ.

Proof: If uncertain sequence $\{\xi_i\}$ completely converges to ξ, it follows from Axiom 3 that

$$M\{\bigcup_{k=i}^{\infty} |\xi_k - \xi| \ge \varepsilon\} \le \sum_{k=i}^{\infty} M\{|\xi_k - \xi| \ge \varepsilon\} \to 0 \tag{23}$$

as $i \to \infty$. Thus, uncertain sequence $\{\xi_i\}$ converges uniformly almost surely to ξ.

Theorem 3.4 Suppose $\xi, \xi_1, \xi_2, \cdots$ are uncertain variables defined on uncertainty space (Γ, L, M). If $\{\xi_i\}$ completely converges to ξ, then $\{\xi_i\}$ converges almost surely to ξ.

Proof: According to Definition 3.2, we have

$$\lim_{i\to\infty} \sum_{k=i}^{\infty} M\{|\xi_k - \xi| \ge \varepsilon\} = 0 \tag{24}$$

It follows from Axiom 3 that

$$M\{\bigcap_{i=1}^{\infty}\bigcup_{k=i}^{\infty}|\xi_k-\xi|\ge\varepsilon\}\le M\{\bigcup_{k=i}^{\infty}|\xi_k-\xi|\ge\varepsilon\} \le\sum_{k=i}^{\infty}M\{|\xi_k-\xi|\ge\varepsilon\}, \quad (25)$$

taking the limitation of $i\to\infty$ on both side of above inequality, we can get

$$M\{\bigcap_{i=1}^{\infty}\bigcup_{k=i}^{\infty}|\xi_k-\xi|\ge\varepsilon\}=0. \quad (26)$$

By Theorem 2.4, we can get the uncertain sequence $\{\xi_i\}$ converges almost surely to ξ.

Example 3.3 Convergence almost surely does not imply complete convergence. For example, take an uncertainty space (Γ,L,M) to be $\{\gamma_1,\gamma_2,\cdots\}$ with $M\{\gamma_i\}=\frac{2i}{4i+1}$ for $i=1,2,\cdots$. Then uncertain variables are defined by

$$\xi_i(\gamma_j)=\begin{cases}2i, & if\ j=i\\ 0, & otherwise\end{cases}, \quad (27)$$

for $i=1,2,\cdots$ and $\xi\equiv 0$. The uncertain sequence $\{\xi_i\}$ converges almost surely to ξ. However, for some small number $\varepsilon>0$, we have

$$M\{|\xi_i-\xi|\ge\varepsilon\}=\frac{2i}{4i+1}\to\frac{1}{2} \quad (28)$$

as $i\to\infty$.

Thus

$$\lim_{i\to\infty}\sum_{k=i}^{\infty}M\{|\xi_k-\xi|\ge\varepsilon\}\ne 0. \quad (29)$$

Therefore, the uncertain sequence $\{\xi_i\}$ does not completely converge to ξ.

Theorem 3.5 Suppose $\xi,\xi_1,\xi_2,\cdots$ are uncertain variables defined on uncertainty space (Γ,L,M). If $\{\xi_i\}$ completely converges to ξ, then $\{\xi_i\}$ converges in measure to ξ.

Proof: Since complete convergence means convergence uniformly almost surely. Then it follows from Theorem 2.6 that uncertain sequence $\{\xi_i\}$ converges in measure to ξ.

Example 3.4 In Example 3.1, uncertain sequence $\{\xi_i\}$ converges in measure to ξ. However, for each $i>1$,

$$\begin{aligned}&\lim_{i\to\infty}\sum_{k=i}^{\infty}M\{|\xi_k-\xi|\ge\varepsilon\}\\ &=\lim_{i\to\infty}\sum_{k=i}^{\infty}M\{\gamma\,|\xi_k(\gamma)-\xi(\gamma)|\ge\varepsilon\}\\ &=\lim_{i\to\infty}\sum_{k=i}^{\infty}\frac{1}{2k}\\ &=\frac{1}{2}\lim_{i\to\infty}\sum_{k=i}^{\infty}\frac{1}{k},\end{aligned} \quad (30)$$

as harmonic series $\sum_{i=1}^{\infty}\frac{1}{i}$ does not converge to 0 when $i\to\infty$, the uncertain sequence $\{\xi_i\}$ does not completely converge to ξ.

Theorem 3.6 Suppose $\xi,\xi_1,\xi_2,\cdots$ are uncertain variables defined on uncertainty space (Γ,L,M). If $\{\xi_i\}$ completely converges to ξ, then $\{\xi_i\}$ converges in distribution to ξ.

Proof: According to Theorem 3.5 and Theorem 2.5, we know that uncertain sequence $\{\xi_i\}$ converges in distribution to ξ.

Example 3.5 For Example 3.2, we know that uncertain sequence $\{\xi_i\}$ converges in distribution to ξ. However, $|\xi_i-\xi|=2a$, for $\gamma=\gamma_1,\gamma_2$, and for some given number $\varepsilon>0$, we have

$$M\{|\xi_i-\xi|\ge\varepsilon\}=1 \quad (31)$$

Then

$$\lim_{i\to\infty}\sum_{k=i}^{\infty}M\{|\xi_k-\xi|\ge\varepsilon\}\ne 0 \quad (32)$$

Thus, the uncertain sequence $\{\xi_i\}$ does not completely converge to ξ.

4 Conclusions

A new concept of convergence (complete convergence) for uncertain sequence was proposed in this paper. Then in the setting of uncertainty theory, we discussed the relationships among these different convergence concepts, which are complete convergence, convergence in p-distance, convergence in measure, convergence in distribution, convergence uniformly almost surely and convergence almost surely. The relationships among uncertain convergence concepts are shown in Figure 1.

Complete Convergence	→	*Convergence Uniformly Almost Surely*	→	*Convergence Almost Surely*
		↓		
Convergence in p – distance	→	*Convergence in Measure*	→	*Convergence in Distribution*

FIGURE 1 Relationships among Convergence Concepts

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