Existence and multiplicity of the solutions for singular fourth-order boundary value problem

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Received 1 June 2014, www.cmnt.lv

Abstract

In this paper, we research the existence and multiplicity of the solution for singular fourth-order boundary value problem: $x^{(4)} = f(t, x(t)), 0 < t < 1$, with the boundary conditions $x(0) = x(1) = x''(0) = x''(1) = 0$. In this singular boundary value, the function has no monotonicity. By using the method of topological degree, we establish solution existence theorem of singular boundary value problem.

Keywords: functional differential equation, singular, positive solutions, fourth-order singular boundary value problem

1 Introduction

The ordinary differential equations with singularity appear in the application of disciplines such as the gas dynamics, fluid mechanics, and the boundary layer theory. The simply supported on both ends of bending elastic beam balance can be described by fourth-order boundary value problems:

$$\begin{cases} x^{(4)} = f\left(t, x(t)\right), 0 < t < 1, \\ x(0) = x(1) = x''(0) = x''(1) = 0. \end{cases} \tag{1}$$

This paper is dedicated singular boundary value problems where the monotonicity of f is removed. By using the method of topological degree [1-4], multiple solution existence theorem of singular boundary value problems is established [5-8], namely f has singularity at $t = 0, t = 1$ and $x = 0$, satisfying (H_1) $f : (0,1) \times (0,\infty) \to [0,+\infty)$ is continuous, $f(\mathrm{t},\mathrm{u}) \le p(\mathrm{t})q(\mathrm{u})$ where $p : (0,1) \to [0,+\infty)$ is continuous, and $q : [0,+\infty) \to [0,+\infty)$ is continuous.

2 Preliminary knowledge and lemma

To state our result, we need some notations. Let $C[0,1], C^2[0,1]$ be Banach space, We denote by $C[0,1]$ the norm as $\|\cdot\|, \|x\| = \max_{0\le t\le 1}|x(\mathrm{t})|$. We take the equivalent norm in $C^2[0,1]$, $\|x\|_2 = \|x\| + \|x''\| = \max_{0\le t\le 1}|x(\mathrm{t})| + \max_{0\le t\le 1}|x''(\mathrm{t})|$.

Suppose $G(\mathrm{t},\mathrm{s}) : [0,1]\times[0,1] \to \mathrm{R}$ be Green function of second order linear boundary value problems $-x'' = 0$, $x(0) = x(1) = 0$, such as:

$$G(\mathrm{t},\mathrm{s}) = \begin{cases} s(1-\mathrm{t}), \ 0 \le s \le t \le 1, \\ t(1-\mathrm{s}), \ 0 \le t \le s \le 1. \end{cases} \tag{2}$$

A is the operation on $C^2[0,1]$:

$$(Ax)(t) = \int_0^1 G(t,\xi)\int_0^1 G(\xi,s) f(s,x(s))dsd\xi . \tag{3}$$

Through the property of the Green function, we can known for $x \in C^2[0,1]$ for $Ax \in C^2[0,1]$ and

$$(Ax)''(t) = -\int_0^1 G(t,s) f(s,x(s))ds . \tag{4}$$

According to Equation (4) and the property of the Green function, we get $(Ax)'' \in C^2[0,1]$, namely $Ax \in C^4[0,1]$ and:

$$(Ax)^{(4)}(t) = f(t,x(t)) . \tag{5}$$

So $A: C^2[0,1] \to C^4[0,1]$ is a continuous map. By compactness of embedded $C^4[0,1] \to C^2[0,1]$, We get $A: C^2[0,1] \to C^2[0,1]$ is a universal continuous map. From the equation (5), $x \in C^2[0,1]$ is the solution of boundary problem(1) if and only if x is the fixed point of A.

Lemma 1: Let X be Banach Space, and $K \subset X$ be defined a cone on X. For $\forall p > 0$, K_p is defined by

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$K_p = \{x \in K | \|x\| < p\}$. Suppose $F : K_p \to K$ be an universal continuous map, when $x \in \partial K_p = \{x \in K | \|x\| = p\}$ there are $Fx \neq x$, then

1) $\|x\| < \|Fx\|, x \in \partial K_p$ containing $i(F, K_p, K) = 0$;

2) $\|x\| > \|Fx\|, x \in \partial K_p$ containing $i(F, K_p, K) = 1$.

Let E be $E = C^2[0,1]$ and K is defined by:

$$K = \left\{ x \in C^2[0,1] \middle| x \geq 0, \min_{t \in J} x(t) \geq \frac{1}{4}\|x\| \right\},$$

where $J = \left[\frac{1}{4}, \frac{3}{4}\right] \subset [0,1]$ then K can be easily proved that K is a cone of E.

Lemma 2: If $A(K) \subset K$ and $A : K \to K$, then A is an universal continuous operation.

Proof: For $\forall t, s \in [0,1]$, there are $G(t,s) \leq G(s,s)$. When $\frac{1}{4} \leq t \leq \frac{3}{4}$, if $s \in [t,1]$, then:

$G(t,s) = t(1-s) \geq \frac{1}{4}(1-s) \geq \frac{1}{4}s(1-s)$; if $s \in [0,t]$, then:

$G(t,s) = s(1-t) \geq \frac{1}{4}s \geq \frac{1}{4}s(1-s)$. Form the above, we have $G(t,s) \geq \frac{1}{4}s(1-s), \quad t \in J, \quad s \in [0,1]$. So:

$$\min_{t \in J}(Ax)(t) = \min_{t \in J} \int_0^1 G(t,\xi) \int_0^1 G(\xi,s) f(s,x(s)) ds d\xi \geq$$
$$\frac{1}{4}\int_0^1 \xi(1-\xi) \int_0^1 G(\xi,s) f(s,x(s)) ds d\xi \geq \frac{1}{4}\|Ax\|.$$

Thus, there are $Ax \in K$, this means $A(K) \subset K$.

The followings prove $A : K \to K$ is an universal continuous operation. Let $B \subset K$ be a bounded set, then there exists $L_1 > 0$, such that $\|x\| \leq L_1$, for $\forall x \in B$. From (H_1), we get:

$$|Ax(t)| < \int_0^1 G(t,\xi) \int_0^1 G(\xi,s) p(\xi) q(x(s)) ds d\xi <$$
$$\max\{q(x) : 0 \leq x \leq L_1\} \int_0^1 G(s,s) \int_0^1 G(s,s) p(s) ds.$$

Those mean $A(B)$ is uniformly bounded.

As, $G(t,s)$ is continues on $[0,1]\times[0,1]$, thus G is uniformly continues. So for all $\forall \varepsilon > 0$, there exists $\exists \delta > 0$, such as when $|t_1 - t_2| < \delta$, we get:

$$|G(t_1,\xi) - G(t_2,\xi)| <$$
$$\varepsilon(\max\{q(x) : 0 \leq x \leq L_1\} \int_0^1 G(s,s) \int_0^1 G(s,s) p(s) ds)^{-1}.$$

Then for all $\forall x \in B$, there are:

$$|Ax(t_2) - Ax(t_1)| \leq \int_0^1 |G(t_2,\xi) - G(t_1,\xi)| \left(\int_0^1 G(s,s) p(s) ds \right)$$
$$\max\{q(x) : 0 \leq x \leq L_1\} d\xi < \varepsilon$$

This means $A(B)$ is relative convergence.

Suppose $x_n, x_0 \in K, x_n \to x_0$, then $\{x_n\}$ is bounded, thus there are

$$|Ax_n(t) - Ax_0(t)| \leq \int_0^1 G(t,\xi) \int_0^1 G(\xi,s) p(\xi) |q(x_n) - q(x_0)| ds d\xi$$

Form continuity of q and Lebesgue dominated convergence theorem, we know $\|Ax_n - Ax_0\| \to 0$. This means $A : K \to K$ is a continuous operation. Then $A : K \to K$ is an universal continuous operation.

3 Main Results

Theorem 1 Suppose f satisfy (H_1) and the following the conditions:

(H_2) $\underline{\lim}_{x \to 0} \inf_{t \in [0,1]} \frac{f(t,x)}{x} = \infty, \underline{\lim}_{x \to \infty} \inf_{t \in [0,1]} \frac{f(t,x)}{x} = \infty$

(H_3) there exists p for $0 \leq x \leq p$, such that $0 \leq q(x) \leq \frac{1}{2}M^{-1}p$, where $0 < M = (\int_0^1 G(t,s) p(s) ds) < +\infty$.

Then there exist two positive solutions x_1, x_2 for the singular fourth-order boundary value problem, which satisfy $0 < \|x_1\|_2 < p < \|x_2\|_2$.

Proof: It is very easy to prove that there exists $\lambda > 0$ satisfying $\frac{1}{4}\lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s) ds > \frac{\|x\|_2}{\|x\|} > 1$. From (H_2) we can know that there exists r satisfying $0 < r < p$, such that if $x < r, t \in [0,1]$ for then $f(t,x) > \lambda x$. When $x \in \partial K_r$, form the definition of K we can know:

$$\left|(Ax)''\left(\frac{1}{2}\right)\right| = \int_0^1 G\left(\frac{1}{2}, s\right) f(s, x(s)) ds >$$
$$\lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) x ds > \frac{1}{4}\lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) ds \|x\| > \|x\|_2,$$

thus, $\|Ax\|_2 > \|Ax\| > \|x\|_2$. So:

$$i(A, K_r, K) = 0. \tag{6}$$

Meanwhile from (H_2) we can know that there exists R satisfying $\bar{R} > 0$, if $x > \bar{R}$, then one has $f(t,x) > \lambda x$, Let $R = \max\{p, 4\bar{R}\}$, if $x \in \partial K_R$, then we have

$$\min_{t \in J} x(t) \geq \frac{1}{4}\|x\| > \bar{R},$$

$$\left|(Ax)''\left(\frac{1}{2}\right)\right| = \int_0^1 G\left(\frac{1}{2}, s\right) f(s, x(s))ds >$$
$$\lambda\int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) xds > \frac{1}{4}\lambda\int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) ds\|x\| > \|x\|_2$$

Thus, there are $\|Ax\|_2 > \|x\|_2$. So:

$$i(A, K_r, K) = 0. \tag{7}$$

On the other hand, from (H_2) we can know if $x \in \partial K_p$, then one have $\|x\|_2 = p$ and:

$$|Ax(t)| \le \int_0^1 G(t,\xi)\int_0^1 G(s,s) f(s, x(s))dsd\xi \le$$
$$\int_0^1 G(t,\xi)\int_0^1 G(s,s) p(s)q(x(s))dsd\xi \le$$
$$\frac{1}{2}\int_0^1 G(s,s)ds\left(\int_0^1 G(s,s)p(s)ds\right)M^{-1}p \le \frac{1}{2}MM^{-1}p = \frac{1}{2}\|x\|_2,$$
$$|(Ax)''(t)| \le \int_0^1 G(s,s) f(s, x(s))ds \le$$
$$\int_0^1 G(s,s)p(s)ds\frac{M^{-1}p}{2} \le \frac{p}{2} = \frac{1}{2}\|x\|_2,$$

So $\|Ax\|_2 = \|Ax\| + \|(Ax)''\| < \frac{1}{2}\|x\|_2 + \frac{1}{2}\|x\|_2 = \|x\|_2$, and

$$i(A, K_p, K) = 0. \tag{8}$$

From Equations (6)-(8), we can know:

$$i(A, K_R \setminus \bar{K}_p, K) = -1, \quad i(A, K_p \setminus \bar{K}_r, K) = 1.$$

Thus, there exist two positive solutions x_1, x_2 of A on $C^2[0,1]$ satisfying $0 < \|x_1\|_2 < p < \|x_2\|_2$.

Theorem 2: Suppose f satisfy (H_2) and the following the conditions:

(H_4) $\overline{\lim}_{x\to 0}\frac{q(x)}{x} < \frac{1}{2}M^{-1}, \overline{\lim}_{x\to\infty}\frac{q(x)}{x} < \frac{1}{2}M^{-1}$,

(H_5) there exists p for $\frac{p}{4} \le x \le p$, such that $f(t,x) > \sigma p$, where $\sigma = \left(\int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)ds\right)^{-1}$,

Then there are two positive solutions x_1, x_2 for the singular fourth-order boundary value problem, which satisfy $0 < \|x_1\|_2 < p < \|x_2\|_2$.

Proof: From (H_5) we can know that there exists r_1 satisfying $\varepsilon: 0 < r_1 < p, 0 < \varepsilon < \frac{M^{-1}}{2}$, such that if $0 < x \le r_1$ then $q(x) \le (\frac{M^{-1}}{2} - \varepsilon)x$, and if $x \in \partial K_{r_1}$, then ones have $\|x\|_2 = r_1$.

$$|Ax(t)| \le \int_0^1 G(t,\xi)\int_0^1 G(s,s) f(s, x(s))dsd\xi \le$$
$$\int_0^1 G(t,\xi)\int_0^1 G(s,s)p(s)q(x(s))dsd\xi \le$$
$$\frac{1}{2}\int_0^1 G(s,s)ds\left(\int_0^1 G(s,s)p(s)ds\right)\left(\frac{1}{2}M^{-1} - \varepsilon\right)r \le$$
$$M(\frac{1}{2}M^{-1} - \varepsilon)r_1\frac{1}{2}r_1,$$
$$|(Ax)''(t)| \le \int_0^1 G(s,s) f(s, x(s))ds \le$$
$$\int_0^1 G(s,s)p(s)ds\left(\frac{1}{2}M^{-1} - \varepsilon\right)r_1 \le$$
$$M\left(\frac{1}{2}M^{-1} - \varepsilon\right)r \le \frac{1}{2}r_1.$$

Thus, $\|Ax\|_2 = \|Ax\| + \|(Ax)''\| < r_1 = \|x\|_2$. So:

$$i(A, K_{r_1}, K) = 0. \tag{9}$$

Meanwhile from (H_4) we can know that there exists r_2, ε_1, satisfying $r_2 > 0, 0 < \varepsilon_1 < 1$, if $x \ge r_2$, then one has $q(x) \le (\frac{M^{-1}}{2} - \varepsilon_1)x$. Let $M_0 = \max\{q(x)| 0 \le x \le r_2\}$, if $\forall x \in [0, +\infty]$, then we have $q(x) \le (\frac{M^{-1}}{2} - \varepsilon_1)x + M_0$. Let $R_2 = \max\{\varepsilon_1^{-1}M_{0,}r_2\}$, if $\forall x \in \partial K_{R_2}$, then we have $\|x\|_2 = R_2$ and:

$$|Ax(t)| \le \int_0^1 G(t,\xi)\int_0^1 G(s,s) f(s, x(s))dsd\xi \le$$
$$\int_0^1 G(t,\xi)\int_0^1 G(s,s)p(s)q(x(s))dsd\xi \le$$
$$\frac{1}{2}\int_0^1 G(s,s)ds(\int_0^1 G(s,s)p(s)ds)\left[\left(\frac{1}{2}M^{-1} - \varepsilon_1\right)x + M_0\right]$$
$$\le \frac{1}{2}\int_0^1 G(s,s)dsM\left[\left(\frac{1}{2}M^{-1} - \varepsilon_1\right)x + M_0\right] \le$$
$$\frac{1}{2}R_2 - M\varepsilon_1 R_2 + MM_0 \le \frac{1}{2}R_2 - MM_0 + MM_0 =$$
$$\frac{1}{2}R_2 = \frac{1}{2}\|x\|_2.$$
$$|(Ax)''(t)| \le \int_0^1 G(s,s) f(s, x(s))ds \le$$
$$\int_0^1 G(s,s)p(s)ds\left[\left(\frac{1}{2}M^{-1} - \varepsilon_1\right)x + M_0\right] \le$$
$$M\left[(\frac{1}{2}M^{-1} - \varepsilon_1)x + M_0\right] \le$$
$$\frac{1}{2}R_2 - MM_0 + MM_0 = \frac{1}{2}R_2 = \frac{1}{2}\|x\|_2.$$

Thus, $\|Ax\|_2 < \|x\|_2$. So:

$$i(A,K_{R_2},K)=0\,. \tag{10}$$

When $\forall x\in\partial K_p, \|x\|_2 = p$, we have

$$\left|(Ax)''\left(\frac{1}{2}\right)\right| = \int_0^1 G\left(\frac{1}{2},s\right) f(s,x(s))ds > \sigma\int_0^1 G\left(\frac{1}{2},s\right)ds\cdot p = p = \|x\|_2\,.$$

So:

$$i(A,K_p,K)=0\,. \tag{11}$$

From (9)-(11), we can know:

$$i(A,K_{R_2}\setminus\bar{K}_p,K)=1,\quad i(A,K_p\setminus\bar{K}_{r_1},K)=-1$$

So there are two positive solutions x_1,x_2 of A on $C^2[0,1]$ satisfying $0<\|x_1\|_2<p<\|x_2\|_2$.

Acknowledgments

The work was supported in part by the National Natural Science Foundation of China (Nos. 71403149).

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