Research of numerical solutions of differential equations model based on the finite element method

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Abstract

Using the finite element method solving a class of second order ordinary differential equations, analyses the two-point boundary value problem of a class of second order ordinary differential equations, through numerical examples to validate its effectiveness.

Keywords: ordinary differential equations, finite element method, two-point boundary value problem

1 Introduction

Finite element method (fem) is a calculating method that booming in the 1960's, it has a wide range of applications, such as elasticity related problem [1-2], related problems in fluid mechanics [3], the heat conduction problem [4-5].

Using the finite element method solves the following two-point boundary value problem of second order ordinary differential equations:

$$\begin{cases} Ly := -\frac{d}{dx}(p(x)\frac{du}{dx}) + r(x)(\frac{du}{dx}) + q(x)u = f(x) \qquad a < x < b & (1) \\ u(a) = u'(b) = 0 & (2) \end{cases}$$

Among them $p(x) \in C'[a,b], r(x), q(x), f(x) \in C[a,b], p(x) \ge p_{\min} > 0, q(x) \ge 0$.

2 Generalized solution

Through the following method, the equation (3) and equation (1) are equivalent, the solution of original problem is called the classical solution, and solution meet the integral form is called the generalized solution. To determine the generalized solution of the new equation is the starting point of the finite element method (fem).

Use it multiply equation (1) both ends, and find the x integrate on [a, b], using part of the integral, then we can get:

$$-p(x)u'(x)\Big|_{x=a}^{x=b} + \int_a^b p(x)u'(x)\phi'(x)dx + \int_a^b r(x)u'(x)\phi(x)dx + \int_a^b q(x)u(x)\phi(x)dx = \int_a^b f(x)\phi(x)dx.$$

By the boundary conditions (2) and $\phi(a) = 0$, then

$$\int_a^b p(x)u'(x)\phi'(x)dx + \int_a^b r(x)u'(x)\phi(x)dx + \int_a^b q(x)u(x)\phi(x)dx = \int_a^b f(x)\phi(x)dx. \quad (3)$$

Then the $u(x)$ is the general solution of boundary problem (1)-(2).

3 Element subdivision and interpolation

Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, element e_i is region $[x_i, x_{i+1}]$, let $u(x_i) = u_i$ in $x = x_i$, u_0 is known, $u_1, \cdots, u_n$ is unknown. In finite element method (fem), when determining the subdivision, determine the specific form of interpolation polynomial on each small unit , and express it through the node function value, that means, in every element e_i, let $u(x) = N_i(x)u_i + M_i(x)u_{i+1} = [N]\{\delta\}e_i, \phi(x) = [N]\{\delta^*\}_{ei}$, and $N_i(x) = \frac{1}{L_i}(x_{i+1} - x), M_i(x) = \frac{1}{L_i}(x - x_i)$ is linear interpolation function.

$L_i = x_{i+1} - x_i, \phi(x)$ is virtual displacement $[N] = (N_i(x), M_i(x)), \{\delta\}_{e_i} = (u_i, u_{i+1})^T$ is nodal displacement in element e_i, $\{\delta^*\}_{e_i} = (u_i^*, u_{i+1}^*)^T$ is node virtual displacement vector in element e_i.

4 Computing units stiffness matrix and load vector

In this paper, the finite element method (fem) is given by specific physical instance and in the engineering, matrix $[K]_{e_i}$ reflects the unit's rigidity, $[K]_{e_i} \cdot \{\delta\}_{e_i} = \begin{pmatrix} -U_i^{e_i} \\ -U_{i+1}^{e_i} \end{pmatrix}$ show that in order to maintain deformation of the unit e_i, two endpoints nodes $x = x_i$, x_{i+1} in unit e_i need external force, to his balance, the two external force is the force

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$-U_i^{e_i}$ and $-U_{i+1}^{e_i}$ through node, and the force known as the equivalent nodal force. While unit load vector $\{F\}e_i$ reflects the effect on the unit e_i displacement distribution of physical strength, its each component shows after the displacement the effect on the two end node $x = x_i, x_{i+1}$ of equivalent strength, therefore, at first, we need to calculate the element stiffness matrix.

Rewrite (3) as

$$\sum_{i=0}^{n-1}\int_{e_i}[p(x)u'(x)\phi'(x)+r(x)u'(x)\phi(x)+q(x)u(x)\phi(x)]dx = \sum_{i=0}^{n-1}\int_{e_i} f(x)\phi(x)dx\ . \tag{4}$$

Using u_i, u_{i+1} and $u_i^*, u_{i=1}^*$ represent $\int_{e_i}(pu'\phi'+ru'\phi+qu\phi)dx, \int_{e_i} f\phi dx$. If a number of a is regarded as a matrix of order, we can see $a^T = a$, then

$$\begin{aligned}\int_{e_i}(pu'\phi'+ru'\phi+qu\phi)dx &= \int_{x_i}^{x_{i+1}}(p(\phi')^T u' + r\phi^T u' + q\phi^T u)dx \\ &= \int_{x_i}^{x_{i+1}} p([B]\{\delta^*\}_{e_i})^T([B]\{\delta\}_{e_i})dx + \int_{x_i}^{x_{i+1}} r([N]\{\delta^*\}_{e_i})^T([B]\{\delta\}_{e_i})dx + \int_{x_i}^{x_{i+1}} q([N]\{\delta^*\}_{e_i})^T([N][\delta]_{e_i})dx \\ &= \{\delta^*\}_{e_i}^T(\int_{x_i}^{x_{i+1}} p[B]^T[B]dx + \int_{x_i}^{x_{i+1}}[N]^T[B]dx + \int_{x_i}^{x_{i+1}} q[N]^T[N]dx)\{\delta\}_{e_i}\, dx \\ &= \{\delta^*\}_{e_i}^T[K]_{e_i}\{\delta\}_{e_i}\end{aligned}\ . \tag{5}$$

$[K]_{e_i}$ is called element stiffness matrix, its concrete form is:

$$[K]_{e_i} = \int_{x_i}^{x_{i+1}}(p[B]^T[B]+[N]^T[B]+q[N]^T[N])dx$$
$$= \begin{pmatrix} \frac{p_i}{L_i^2}+\frac{r_i}{L_i}\int_{x_i}^{x_{i+1}} N_i dx + \int_{x_i}^{x_{i+1}} qN_i^2 dx & -\frac{p_i}{L_i^2}+\frac{r_i}{L_i}\int_{x_i}^{x_{i+1}} N_i dx + \int_{x_i}^{x_{i+1}} qN_iM_i dx \\ -\frac{p_i}{L_i^2}+\frac{r_i}{L_i}\int_{x_i}^{x_{i+1}} N_i dx + \int_{x_i}^{x_{i+1}} qN_iM_i dx & \frac{p_i}{L_i^2}+\frac{r_i}{L_i}\int_{x_i}^{x_{i+1}} N_i dx + \int_{x_i}^{x_{i+1}} qM_i^2 dx \end{pmatrix}.$$

Among it, $p_i = \int_{x_i}^{x_{i+1}} p(x)dx$.

Make the element stiffness matrix as $[K]_{e_i} = \begin{pmatrix} k_{i,i}^{e_i} & k_{i,i+1}^{e_i} \\ k_{i+1,i}^{e_i} & k_{i+1,i+1}^{e_i}\end{pmatrix}$, then stiffness coefficient $k_{j,m}^{e_i}$ is

$$k_{i,i}^{e_i} = \frac{p_i}{L_i^2}+\frac{r_i}{L_i}\int_{x_i}^{x_{i+1}} N_i dx + \int_{x_i}^{x_{i+1}} qN_i^2 dx$$

$$k_{i,i+1}^{e_i} = k_{i+1,i}^{e_i} = -\frac{p_i}{L_i^2}+\frac{r_i}{L_i}\int_{x_i}^{x_{i+1}} N_i dx + \int_{x_i}^{x_{i+1}} qN_iM_i dx,$$

$$k_{i+1,i+1}^{e_i} = \frac{p_i}{L_i^2}+\frac{r_i}{L_i}\int_{x_i}^{x_{i+1}} N_i dx + \int_{x_i}^{x_{i+1}} qM_i^2 dx\ . \tag{6}$$

$$\int_{e_i} f\phi dx = \int_{e_i}\phi^T f dx$$
$$= \int_{x_i}^{x_{i+1}}([N]\{\delta^*\}_{e_i})^T f dx = \{\delta^*\}_{e_i}^T\{F\}_{e_i}$$

$\{F\}_{e_i}$ is called Unit load vector, its concrete form is:

$$\{F\}_{e_i} = \int_{x_i}^{x_{i+1}}[N]^T f dx = \begin{pmatrix}\int_{x_i}^{x_{i+1}} N_i f dx \\ \int_{x_i}^{x_{i+1}} M_i f dx\end{pmatrix}.$$

Expressed $\{F\}_{e_i}$ as $\{F\}_{e_i} = \begin{pmatrix} F_i^{e_i} \\ F_{i+1}^{e_i}\end{pmatrix}$, then, unit load coefficient is $F_i^{e_i} = \int_{x_i}^{x_{i+1}} N_i f dx, F_{i+1}^{e_i} = \int_{x_i}^{x_{i+1}} M_i f dx$.

5 Total stiffness matrix and the total load vector

Expand $[K]_{e_i}, \{F\}_{e_i}, \{\delta\}_{e_i}$ and $\{\delta^*\}$ for n + 1 order matrix and the n + 1 d vector, put (5) and (6) into (4), then

$$\begin{aligned}&\{\delta^*\}^T(\sum_{i=0}^{n-1}[K]_{e_i})\{\delta\} = \{\delta^*\}^T(\sum_{i=0}^{n-1}\{F\}_{e_i}) \\ &\{\delta^*\}^T([K]\{\delta\}-\{F\}) = 0\end{aligned}\ . \tag{7}$$

Among them, $[K] = \sum_{i=0}^{n-1}[K]_{e_i}, \{F\} = \sum_{i=0}^{n-1}\{F\}_{e_i}$ as the total stiffness matrix and the total load vector respectively.

6 Constraint handling

As on the endpoint x = 0, $u(x)$ satisfy the boundary conditions (2), therefore $u_0 = 0$, in addition $\phi(0) = 0$, so

$u_0^* = 0$, and (7) has the form $\left\{\tilde{\delta}^*\right\}^T ([\tilde{K}]\{\delta\} - \{\tilde{F}\}) = 0$, $\left\{\tilde{\delta}^*\right\}$, $\left\{\tilde{F}\right\}$ are respectively after the first element with 0 instead of n+1-dimensional vector in $\left\{\delta^*\right\}$, $\left\{F\right\}$, $\left[\tilde{K}\right]$ is the element in the first row and the first column in addition to the diagonal to 1, use 0 instead of the back rest of the n+1 matrix. Due to the arbitrary of $\left\{\tilde{\delta}^*\right\}$, we can obtain:

$$\left[\tilde{K}\right]\{\delta\} = \left\{\tilde{F}\right\}. \tag{8}$$

From the change in the overall stiffness matrix $[K]$ for the first line in the first column elements, forming process of $\left[\tilde{K}\right]$, known as the constraint processing. Main purpose is to require the desires of function satisfy the boundary constraint conditions $u_0 = 0$ on boundaries. After dealing with the constraints, a new overall stiffness matrix $\left[\tilde{K}\right]$ from positive semi-definite to positive definite. In fact, because: $\left\{\tilde{\delta}\right\}^T [K]\{\delta\} = (0, u_1, \cdots, u_n)$ $[K](0, u_1, \cdots, u_n)^T \geq 0$

So $\left[\tilde{K}\right]$ is symmetric positive definite, so a linear equations (8) solution is unique. Because above is our whole process in solving the equation (3), and because equation (3) and equation (1) are equivalent, so here equation (8) is the solution of the original problem (1) what we want.

7 A Numerical example and conclusion

Consider the following second order ordinary differential equation with two-point boundary value problems:

$$\begin{cases} -(x^2 u'(x))' + 2u(x) = 2x^2, 0 < x < 1 \\ u(0) = u'(1) = 0 \end{cases}.$$

Compare with equation (1), this example as $p(x) = x^2$, $r(x) = 0$, $q(x) = 2$, $f(x) = 2x^2$, now the interval [0, 1] is divided into four units, the five coordinates of nodes are $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}, x_4 = 1$ respectively.

Stiffness matrix:

$$[K]_{e_i} = \begin{pmatrix} 4(x_{i+1}^2 - x_{i+1}x_i + x_i^2) & \frac{4}{3}(x_{i+1}^2 + x_i^2) - \frac{20}{3}x_{i+1}x_i \\ \frac{4}{3}(x_{i+1}^2 + x_i^2) - \frac{20}{3}x_{i+1}x_i & 4(x_{i+1}^2 - x_{i+1}x_i + x_i^2) \end{pmatrix}.$$

Unit load vector:

$$[F]_{e_i} = \begin{pmatrix} \frac{2}{3}x_{i+1}(x_{i+1}^2 + x_{i+1}x_i + x_i^2) - \frac{1}{2}(x_i + x_{i+1})(x_i^2 + x_{i+1}^2) \\ \frac{1}{2}(x_i + x_{i+1})(x_i^2 + x_{i+1}^2) - \frac{2}{3}x_{i+1}(x_{i+1}^2 + x_{i+1}x_i + x_i^2) \end{pmatrix}.$$

The overall stiffness matrix is $[K] = \sum_{i=0}^{3}[K]_{e_i}$, total load vector is $[F] = \{F\} = \sum_{i=0}^{3}\{F\}_{e_i}$, the exact solution is $u(x) = x - \frac{1}{2}x^2$, the numerical results are shown in Table 1.

TABLE 1 Numerical result

node	Exact solution	Approximate solution	Absolute error	Relative error
1/4	0.21875	0.21637	2.38E-03	1.088E-02
1/2	0.375	0.37476	2.4E-04	6.4E-04
3/4	0.46875	0.46881	6.0E-05	1.28E-04
1	0.5	0.52475	2.475E-02	4.95E-02

This paper discusses the application of finite element method for solving a class of second order ordinary differential equation numerical solution, analyses a class of second order ordinary differential equation with two-point boundary value problem, the effectiveness of finite element method is verified by a numerical example for solving a class of second order ordinary differential equation numerical solutions algorithm.

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