Continuous-time optimal portfolio model with mean-reverting process

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Abstract

This paper studies a continuous-time portfolio optimization problem. It is proposed a simple but powerful approximation approach that is both accurate and computationally efficient for the terminal expectation of the investors with mean-reverting process, which is different from the existing literatures that apply the dynamic programming method. Numerical examples illustrate the computational efficiency and accuracy of our approach when compared with results from Monte Carlo (MC) simulations.

Keywords: Continuous-time portfolio, Mean-reverting process, Optimization; Monte Carlo

1 Introduction

It was pioneered by Markowitz to propose the mean-variance(M-V) framework, which has been playing a cornerstone role in the theory of portfolio selection [1]. Numerous scholars have extended Markowitz's single period portfolio selection into the counterparts of multi-period and continuous-time. Mossin (1968) [2] considered optimal multi-period portfolio selection by using dynamic programming approach. Hakansson (1971) [3] presented the multi-period mean-variance model using a general portfolio theory. Wu (2014) [4] considered a multi-period mean-variance portfolio selection when the time horizon is assumed to be stochastic and depends on the market states. For more detailed discussion on the subject of dynamic portfolio selection, it is referred to [5-8]. In continuous-time version of dynamic portfolio selection, Yao [9] investigated a continuous-time mean-variance portfolio selection problem with multiple risky assets using the Lagrange duality method and the dynamic programming approach, and derived explicit closed-form expressions of efficient frontier. Yong [10] considered a continuous-time optimal consumption and portfolio selection model with voluntary retirement using the dynamic programming method to derive the optimal strategies in closed-form. Holger [11] studied constrained portfolio problems and solved the problems by dynamic programming. The standard approach to solve the dynamic portfolio optimization problem is martingale method, which was developed by Karatzas et.al (1987) [12] and Cox and Huang (1989) [13]. Because dynamic programming method to solve the problem gives easy access to the value function, many scholars pay much attention to it.

To the best of our knowledge, all the existing literatures about continuous-time portfolio selection model are solved by analysed method, there has been few literatures on approximation method. In fact, for certain investment, all the strategies are not accurate, because the finance market-self is not certain. Therefore, it is reasonable to use approximation method to solve the problem. To deal with uncertain, numerous research on fuzzy mathematic method [14-15]. This paper proposes a mean-reverting process to approximate the distribution of a weighted sum of correlated assets.

The paper is organized as follows. Section 2 gives model formulations, including the mean-reverting process and the two steps of approximation. Section 3 describes the optimal portfolio selection model. Numerical examples, illustrating both the computational efficiency and accuracy of our method are presented in section 4. Section 5 gives concluding remarks.

2 Model formulations

It is considered a portfolio consisting of m assets with price $S_i(t), i=1,2\cdots m$, which are described as the following stochastic differential equation,

$$\begin{cases} dS_i(t)=\mu_i(t)S_i(t)dt+\sigma_i(t)S_i(t)dz_i t \\ S_i(0)=s_{i0} \end{cases}, \forall t\in[0,T], \quad (1)$$

where $\{z_i(t), t>0\}$ are standard Brownian motions, and $\mu_i(t)$, $\sigma_i(t)$ are respectively the appreciation rate and vitality rate of asset i. It is assumed that $\mathrm{cov}(dz_i, dz_j)=\rho_{ij}dt$, ρ_{ij} denote the correlation coefficients between $z_i(t)$ and $z_j(t)$.

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The portfolio via $S(\cdot)$ can be expressed by $\sum_{i=1}^{n}\omega_i S_i(t)$, ω_i is the weight of asset i. As we know, a weighted sum of non-independent lognormals does not have an explicit analytical expression. One can use Monte Carlo simulation techniques to obtain a numerical estimation. Other techniques take account in approximating. For instance, it is used the lognormal distribution for approximating the sum of lognormals. But it leads poor results. Another different methods can see to Albrecher (2004), Curran (1994), and Kawai (2003) [16-18].

In this paper, we present an approximate method of mean-reverting prices. Therefore, it is necessary to introduce it firstly.

2.1 MEAN-REVERTING PROCESS

According to Bos [19], the process x_i follows the exponential Ornstein – Uhlenbeck (EOU) model

$$\begin{cases} dx_i(t)=k_i(t)\left(\gamma_i(t)-x_i(t)\right)dt+v_i(t)x_i(t)dz_i(t), \forall t\in[0,T], \\ x_i(0)=x_{i0} \end{cases} \quad (2)$$

where the parameters $v_i(t)$, $\gamma_i(t)$ and $k_i(t)$ are determined following Duffie and Richardson [20].

A direct calculation yields the first and second moments of $x_i(t)$ are

$$Ex_i(t)=e^{-\int_0^t 2k_i(s)ds}\left(x_{i0}+\int_0^t k_i(s)\gamma_i(s)e^{\int_0^s k_i(u)du}ds\right), \quad (3)$$

$$Ex_i^2(t)=e^{\int_0^t\left(v_i^2(s)-2k_i(s)\right)ds}\int_0^t v_i^2(s)\left(x_{i0}+\int_0^s k_i(u)\gamma_i(u)e^{\int_0^u k_i(v)dv}du\right)^2 e^{-\int_0^s v_i^2(u)du}ds. \quad (4)$$
$$+e^{-\int_0^t 2k_i(s)ds}\left(x_{i0}+\int_0^t k_i(s)\gamma_i(s)e^{\int_0^s k_i(u)du}ds\right)^2$$

Let the portfolio $X(t)=\omega x(t)$, and suppose it follows a mean-reverting process, and let $x_{i0}=S_{i0}$. It means that the mean at the time points of small intervals is with reverting.

2.2 THE APPROXIMATION IN THE FIRST STEP

Suppose the assets prices follow (1), in every small interval $[t\ ,t\,]$, $l=1,2\cdots n$.

$$\begin{cases} \mu_i(t)=\mu_{il} \\ \sigma_i(t)=\sigma_{il} \end{cases}, \forall t\in[t_{l-1},t_l], l=1,2\cdots n. \quad (5)$$

To calculate of (1) in $[t_{l-1},t_l]$ yields

$$\begin{cases} ES_i(t_l)=ES_i(t_{l-1})e^{\mu_{il}\Delta t_l} \\ ES_i^2(t_l)=ES_i^2(t_{l-1})e^{\left(\sigma_{il}^2+2\mu_{il}\right)\Delta t_l} \end{cases}. \quad (6)$$

So

$$\begin{cases} \mu_{il}=\dfrac{1}{\Delta t_l}\ln\dfrac{ES_i(t_l)}{ES_i(t_{l-1})} \\ \sigma_{il}=\sqrt{\dfrac{1}{\Delta t_l}\left[\ln\dfrac{ES_i(t_l)^2}{ES_i(t_{l-1})^2}-2\ln\dfrac{ES_i(t_l)}{ES_i(t_{l-1})}\right]} \end{cases}. \quad (7)$$

The first step approximation is that $S_i(\cdot)\approx X(\cdot)$ in every sufficiently small interval $[t_{l-1},t_l]$, and let the first and second moments of $S_i(t)$ is the same as $x_i(t)$ at point t_{l-1} and t_l. We obtain

$$\begin{cases} \mu_{il}=\dfrac{1}{\Delta t_l}\ln\dfrac{Ex_i(t_l)}{Ex_i(t_{l-1})} \\ \sigma_{il}=\sqrt{\dfrac{1}{\Delta t_l}\left[\ln\dfrac{Ex_i(t_l)^2}{Ex_i(t_{l-1})^2}-2\ln\dfrac{Ex_i(t_l)}{Ex_i(t_{l-1})}\right]} \end{cases}. \quad (8)$$

2.3 THE APPROXIMATION IN THE SECOND STEP

The second step approximation is $X(t)\approx\sum_{i=1}^{m}\omega_i S_i(t)$. A direct calculation yields

$$\begin{cases} EX(T)\approx\sum_{i=1}^{m}\omega_i x_{i0}\exp\left(\sum_{i=1}^{m}\mu_{il}\Delta t_l\right) \\ EX^2(T)\approx\sum_{i=1}^{m}\sum_{i=1}^{m}\omega_i\omega_j x_{i0}x_{j0}\exp\left(\sum_{i=1}^{m}\left(\mu_{il}+\mu_{jl}+\rho_{ij}\sigma_{ij}\right)\Delta t_l\right) \end{cases}. \quad (9)$$

To classify the proposed approximation method, it is summarized the procedure as follows:

- Divide the time period $[0,T]$ into finitely many small intervals $[t_{l-1},t_l]$ and calculate the first and second moments according to (3), (4).
- Substitute the first and second moments to (8), yield μ_{il} and σ_{il}.
- Substitute μ_{il} and σ_{il} to (2.9) obtain the first and second moments of the approximated portfolio at the end of the investment horizon.

3 The optimal portfolio selection model

In order to show the effectiveness, we only consider the simply case. The problem for an investor is to find the optimal strategy to minimize the variance while attaining a given level of the expected wealth.

$$\begin{cases} \min\limits_{\omega} Var\left(X\left(T\right)\right) \\ s.t \;\; E\left(X\left(T\right)\right)=u \end{cases}, \qquad \text{(M1)}$$

where u is a pre-given constant, representing the expected value level, which the investor requires to achieve and $Var\left(X\left(T\right)\right)=E\left(X-EX\right)^2=EX^2\left(T\right)-\left(EX\left(T\right)\right)^2$, $EX^2\left(T\right), EX\left(T\right)$ are obtained from (9).

According to the optimal portfolio model (M1), the problem can be deal with by the Lagrange method. Introducing the Lagrange multiplier λ leads to the following problem

$$\min E\left(X\left(T\right)-u\right)^2+2\lambda\left(EX\left(T\right)-u\right). \qquad \text{(M2)}$$

Let $\pi\left(T\right)=g\left(x\left(T\right),\lambda\right)$ be the optimal solution of the Lagrangian problem (M2) and $G\left(x_0,\lambda\right)$ be the optimal value. According to the Lagrange duality theory, if λ^* satisfies $\max\limits_{\lambda} G\left(x_0,\lambda\right)$, then $\pi^*\left(t\right)=g\left(x\left(t\right),\lambda^*\right)$ is the optimal shares of (M1) and $G\left(x_0,\lambda^*\right)$ is its optimal value. Problem (M2) is equivalent to $\min E\left(x\left(T\right)+a\right)^2$, where $a=\lambda-u$.

4 Numerical examples and applications

In this subpart, we illustrate the accuracy and computational efficiency from two-asset and four assets experiments, and compare our results with MC simulations.

- Two-asset Case

Time to maturity $T=1$ year, risk-free rate 5% per annum, $\nu_1=\nu_2=20\%$, $S_{10}=x_{10}=50,$ $S_{20}=x_{20}=50,$ $\rho=0.5$, $\gamma_1=x_{10}e^{5\%T}=52.5636,$ also so, $\gamma_2=x_{20}e^{5\%T}=52.5636,$ $k_1=k_2=\frac{1}{3},$ $u=8\%$.

Following the step of part 2, solving (M1), the Monte Carlo simulation results of portfolio weights is $\omega_1=0.3891, \omega_2=0.6109$, and the results from the approximation method proposed in our study is $\omega_1=0.3255, \omega_2=0.6755$.

- Four-asset Case

Time to maturity $T=1$ year, risk-free rate 5% per annum, $\nu_1=\nu_2=\nu_3=\nu_4=20\%$, $S_{10}=x_{10}=50,$ $S_{20}=x_{20}=50,$ $S_{30}=x_{30}=50,$ $S_{40}=x_{40}=50,$ $k_1=k_2=k_3=k_4=\frac{1}{3},$ $u=8\%$.

Following the step of part 2, solving (M1), the Monte Carlo simulation results of portfolio weights is $\omega=0.0414, \omega=0.2782, \omega=0.6351, \omega=0.0453$, and the results from the approximation method proposed in our study is $\omega_1=0.0527, \omega_2=0.3181, \omega_3=0.5324, \omega_4=0.0968$.

From the compared results, we find that the portfolio shares from our model and Monte Carlo simulation is nearly the same.

5 Conclusion

Different from the general dynamic approach, we propose the mean-reverting process to approximate the weight sum of assets, and extend the discrete portfolio to the continuous-time case. The optimal portfolio selection model aims to minimize the risk described as VaR meanwhile to get the given expectation is presented. Numerical example results for two-asset and four-asset case shows the accuracy and computational efficiency compared to Monte Carlo simulation results. It provides a new way to deal with the continuous-time portfolio selection.

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