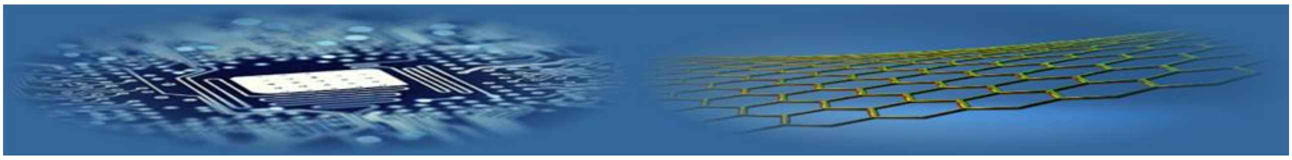

Powerdomains and modality, revisited with detailed proofs

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Abstract

We give direct detailed proofs for the connection between powerdomains and logic models which can be made about nondetermin- istic computations. In the proceeding of proofs, we prove some algebraic properties of them at the same time. Meanwhile, we take up some trick for constructing the finite branching tree, which can also be used into the other areas.

Keywords:

Power domain
Nondeterministic computation
Algebraic properties

1 Introduction

Powerdomains were originally proposed to model the semantics of nondeterministic programming languages [1, 2]. They can be taken as the domain analogues of powersets with elements which represent the sets of different courses a nondeterministic computation can follow. Winskel gives a simple connection between powerdomains and modal assertions that can be made about nondeterministic computations in [1]. He considers three kinds of powerdomains: the Smyth powerdomain, the Hoare powerdomain and the Plotkin powerdomain. Two kinds of modal operators are taken: ¤ for "inevitable" and § for "possible". It is shown in a precise sense how the Smyth powerdomain is built up from assertions about the inevitable behaviour of of a nondeterministic computation, the Hoare powerdomain is built up from assertions about the possible behaviour of of a nondeterministic computation, while the Plotkin powerdomain is built up from both kinds of assertions taken together. In [3], the detailed proofs are not given. It is also a little fuzzy to understand the sketch of proofs. We give here detailed direct proofs of thee results. On the way, we establish some algebraic properties of powerdomains and of nondeterministic computations. In particular, we spell out the construction of nondeterministic computations associated to the elements of the respective powerdomains.

We present the preliminaries on powerdomains and nondeterministic computations. To know more, see [4].

We first give some knowledge of domain theory. Let D be a partial order. A $directed\ set$ of $(D,\leq)$ is a non-null subset S(µ D) such that $\forall s,t\in S\ \exists u\in S\ s.t.\ s\leq u\ \&\ t\leq u$. A $left\text{-}closed$ of (D;·) is a subset A(µ D) such that $\forall a,b\in D.a\leq b\in A\Rightarrow a\in A$. An $ideal$ of $(D,\leq)$ is a non-null subset $A(\subseteq D)$ such that A is a directed and left-closed set. A $complete\ partial\ order$($c.p.o.$ for short) is a partial order $(D,\leq)$ which has a least element $\perp$ and all least upper bounds of directed subsets. A partial order $(D,\leq)$ which has a least element $\perp$ and all least upper bounds of directed subsets. A $finite$ element of a $c.p.o.(D,\leq)$ is an element $x\in D$ such that for any directed subset $S\subseteq D$ when $x\leq \sqcup S$ there is an $s\in S$ such that $x\leq s$. We write D^0 for the set of finite elements of D. Intuitively, the finite elements are that information which a computation can realize in finite time. The $c.p.o.$ $(D,\leq)$ is $algebraic$ if and only if for all $x\in D$ we have$x=\sqcup\{e\leq x\mid e\in D^0\}$. The $c.p.o.$ $(D,\leq)$ is said to be countably algebraic or simply ω-algebraic if and only if D is algebraic and D^0 is countable.

In the sequel, $(D,\leq)$ always stands for an ω-algebraic $c.p.o.$. Let $M[D]$ be the set of non-null finite subsets of D^0. There are three natural ways to preorder $M[D]$. We consider these three kinds of order: for A, B in $M[D]$, write

$$\begin{aligned} A\preccurlyeq_S B &\iff \forall b\in B\ \exists a\in A\ s.t.\ a\leq b\\ A\preccurlyeq_H B &\iff \forall a\in A\ \exists b\in B\ s.t.\ a\leq b\\ A\preccurlyeq_P B &\iff A\preccurlyeq_S B\ \&\ A\preccurlyeq_H B \end{aligned},$$

where $\preccurlyeq_S$, $\preccurlyeq_H$ and $\preccurlyeq_P$ are called $the\ Smyth\ order$, $the\ Hoare\ order$ and $the\ Plotkin\ order$ respectively.

There is a standard way to get an algebraic domain from a preorder with least element, often called $completion\ by\ ideals$ [5]. Let $(P,\leq)$ be a preorder with a least element $\perp$. $I(P)$ is the set of ideals of P. It is easy to prove that $(I(P,\leq),\subseteq)$ is an algebraic domain, with finite elements $\{q\in P\mid q\leq p\}$ for $p\in P$. In this way, we can obtain three different powerdomains by completing by ideals the three preorder $\preccurlyeq_S,\preccurlyeq_H,\preccurlyeq_P$ on $M[D]$. We name them respectively *the Smythpowerdomain*, *the Hoare powerdomain*, and *the Plotkin powerdomain*.

$$\begin{aligned} (P_0[D],\leq_0) &\triangleq (I(M[D],\preccurlyeq_S),\subseteq)\\ (P_1[D],\leq_1) &\triangleq (I(M[D],\preccurlyeq_H),\subseteq)\\ (P_2[D],\leq_2) &\triangleq (I(M[D],\preccurlyeq_P),\subseteq). \end{aligned}$$

Now, we define the notion of nondeterministic D-computation. A tree is $finitely\ branching$ if it has a finite number of branches at each fork.

A nondeterministic D-computation has the form $(T,\rightarrow,val)$ where $(T,\rightarrow)$ is a finitely branching tree and

val is a map to D^0 such that $\forall\, t,t' \in T,\ t \to t' \Longrightarrow val(t) \le val(t')$.

A $branch$ is a sequence $t_0, t_1, \cdots, t_n, \cdots$ where t_0 is the root node and $t_n \to t_{n+1}$ for each $n+1$ at which the sequence is defined. By a maximal branch of $(T, \to)$ we mean a branch which is either infinite, or finite of the form $t_0, t_1, \cdots, t_n$ with $t_n \not\to$.

2 The Smyth Powerdomain

We define a modal logic L_0 including D^0, for all $s \in L_0$, s is any formula built by the following syntax; $s ::= a \mid \Box s \mid s \vee s \quad (a \in D^0)$.

Let $(T, \to, val)$ be a nondeterministic D-computation. Define $\models_T$ to be the least relation included in $T \times L_0$ such that

$$\begin{array}{lll} t \models a & if & a \le val(t) \\ t \models (s \vee s') & if & t \models s \ \ or \ \ t \models s' \\ t \models \Box s & if & t \models s \ \ or \ \ (\forall t', t' \to t \Longrightarrow t' \models \Box s) \end{array}$$

Definition 1. Let T be a finitely branching tree. A coupe of T is a finite set C of the nodes of T such that

$$((t_1, t_2 \in C \ \&\ val(t_1) \le val(t_2)) \Rightarrow t_1 = t_2) \\ \text{and } (\forall s \in T \exists t \in C \ s.t.\ val(s) \le val(t) \\ or\ val(t) \le val(s))$$

Another way to define satisfaction for $\Box$-statements is as follows.

Lemma 1. *Let T be a finitely branching tree with the root t. We have,*

$$t \models \Box s \Leftrightarrow \exists C \ (C \text{ is a coupe of } T)\ s.t.\ (\forall t' \in C, t' \models s)$$

Proof. Suppose that $t \models \Box s$. To prove it, we are only allowed to use the following rules,

$$\frac{t \models s}{t \models \Box s} \quad (1) \qquad\qquad \frac{\forall t', t \to t', t' \models \Box s}{t \models \Box s} \quad (2)$$

The basic case is that from $t \models_T s$ we can get $t \models_T \Box s$. Let C be $\{t\}$. It is clear that C is a coupe, and $t \models s$. The other case is that if for all t' satisfying $t \to t'$, we have $t' \models \Box s$ then we can get $t \models \Box s$. By the assumption of induction, for any $t' \models \Box s$, there exists a coupe $C_{t'}$ such that $\forall t'' \in C_{t'}, t'' \models s$. Let C be the disjoin union of the coupes $C_{t'}$s. Then it is easy to prove that C is also a coupe of T and $\forall t'' \in C, t'' \models s$.

On the other hand, suppose that there is a coupe C of T such that $\forall t'' \in C, t'' \models s$. If $t \in C$, then $t \models \Box s$. If $t \notin C$, for any t' satisfying $t \to t'$.

Lemma 2. Let $T = (T, \longrightarrow, val)$ be a nondeterministic D-computation with root node t. Write $\models_T s$ for $t \models_T s$. Let $\mathbf{T}$ be the class of nondeterministic D-computations. Define $s \equiv s'$ iff $(\models_T s \Leftrightarrow \models_T s', \forall T \in \mathbf{T})$.

We have

(1) $s \vee (s' \vee s'') \equiv (s \vee s') \vee s''$;

(2) $s \vee s' \equiv s' \vee s$;

(3) $\Box(\Box s) \equiv \Box s$;

(4) $\Box(s \vee \Box s') \equiv \Box(s \vee s')$;

(5) $s \equiv s' \Longrightarrow \Box s \equiv \Box s'$;

(6) $s \equiv s' \Longrightarrow s \vee s'' \equiv s' \vee s''$;

Proof (1), (2), (5), (6) It is clear

(3) For any $T \in \mathbf{T}$, if $\models_T \Box s$, then $\models_T \Box(\Box s)$. On the other hand, we assume that $\models_T \Box(\Box s)$, then for every maximal branch φ in the subtree out of *t*, there is a node t' which satisfies $\Box s$. Let λ be the finite branch from t to t'. Define that $\varphi - \lambda$ as φ from which λ has been cut. It is clear that $\varphi - \lambda$ is a maximal branch in the subtree out of t'. So $\varphi - \lambda$ has a node t'' which satisfies *s*, that is, φ has the node t'' which satisfies *s*. By lemma 1, $\models_T \Box s$.

(4) For any $T \in \mathbf{T}$, if $\models_T \Box(s \vee s')$, then for every maximal branch in the subtree out of *t*, there is a node t' which satisfies $s \vee s'$, that is, $t' \models_T s$ or $t' \models_T s'$. It follows that $t' \models_T s$ or $t' \models_T \Box s'$, then $t' \models_T s \vee \Box s'$. So $\models_T \Box(s \vee \Box s')$. On the other hand, if $\models_T \Box(s \vee \Box s')$, then for every maximal branch φ in the subtree out of *t*, there is a node t' which satisfies $\models_T s \vee \Box s'$, that is, $t' \models_T s$ or $t' \models_T \Box s'$. (i) If $t' \models_T s$, then $t' \models_T s \vee s'$; (ii) If $t' \models_T \Box s'$, let λ be the branch from t to t', then $\varphi - \lambda$ is a maximal branch in the subtree out of t', so $\varphi - \lambda$ has a node t'' which satisfies s', hence $t'' \models_T s \vee s'$. Then φ has the node which satisfies $s \vee s'$. Therefore, $\models_T \Box(s \vee s')$.

Such properties make us can get the normal forms of the logic model L_0.

Lemma 3. $\Box s \in L_0$ is $\equiv -$ equivalent a normal form $\Box(a_0 \vee \cdots \vee a_n)$ for some $a_0, \cdots, a_n \in D^0$.

Proof. By definition of L_0, s is any formula built with the following syntax, $s ::= a \mid \Box s \mid s \vee s$.

It suffices to consider the following cases;

The basic case is that *s* = *a*. Then $\Box s = \Box a$, that is, $\Box s$ is $\equiv -$ equivalent the normal form $\Box a$. Another case is that $s = \Box s'$. Suppose that $\Box s'$ is $\equiv -$ equivalent a normal form $\Box(a_0 \vee \cdots \vee a_n)$. Then $\Box s = \Box(\Box s') \equiv \Box(\Box(a_0 \vee \cdots \vee a_n)) \equiv \Box(a_0 \vee \cdots \vee a_n)$ by the lemma 2. Another case is that $s = s' \vee s''$. Suppose that $\Box s'$ is $\equiv -$ equivalent a normal form $\Box(a_0 \vee \cdots \vee a_n)$ and s'' is $\equiv -$ equivalent a normal form $\Box(b_0 \vee \cdots \vee b_m)$. Then $\Box s = \Box(s' \vee s'') \equiv \Box(s' \vee \Box s'') \equiv \Box(\Box s'' \vee s') \equiv \Box(\Box s'' \vee \Box s') \equiv \Box(\Box(b_0 \vee \cdots \vee b_m) \vee \Box(a_0 \vee \cdots \vee a_n)) \equiv \Box(\Box(b_0 \vee \cdots \vee b_m) \vee (a_0 \vee \cdots \vee a_n)) \equiv \Box((a_0 \vee \cdots \vee a_n) \vee \Box(b_0 \vee \cdots \vee b_m)) \equiv \Box(a_0 \vee \cdots \vee a_n \vee b_0 \vee \cdots \vee b_m)$ by lemma 2.

We are interested in the statements which are inevitably true. Define the following set of assertions with nondeterministic computations.

Definition 2. *Let $T = (T, \to, val)$ be a nondeterministic D-computation with root node t. Define*

$$V_0(T) = \{\Box s \in L_0 \mid \models_T \Box s\}$$

Based on this assertions, we define an obvious preorder on nondeterministic computations

$$T \precsim_0 T' \Longleftrightarrow V_0(T) \subseteq V_0(T')$$

Quotienting the preorder $\precsim_0$ on nondeterministic computations by the equivalence $\simeq_0 \triangleq \precsim_0 \cap \precsim_0{}^{-1}$, we obtain the Smyth powerdomain by th eorem 1. Before we prove that, we give some algebraic properties of nondeterministic computations.

Let $V_0'(T) = \{\Box(a_0 \vee \cdots \vee a_n) \mid \models_T \Box(a_0 \vee \cdots \vee a_n), a_0, ..., a_n \in D^0\}$.

Lemma 4. Let T and T' be nondeterministic computation. t is the root of T and T' is the root of T', we have $V_0(T) \subseteq V_0(T') \Longleftrightarrow V_0'(T) \subseteq V_0'(T')$.

Proof. Firstly, we assume that $V_0(T) \subseteq V_0(T')$. Let $\Box(a_0 \vee \cdots \vee a_n) \in V_0'(T)$. Then $t \models_T \Box(a_0 \vee \cdots \vee a_n)$, because $\Box(a_0 \vee \cdots \vee a_n) \in V_0'(T) \in V_0(T)$, then $\Box(a_0 \vee \cdots \vee a_n) \in V_0(T')$ since $V_0(T) \subseteq V_0(T')$. Therefore $t' \models_{T'} \Box(a_0 \vee \cdots \vee a_n)$, and $\Box(a_0 \vee \cdots \vee a_n) \in V_0'(T')$.

On the other hand, we assume that $V_0'(T) \subseteq V_0'(T')$. Let $\Box s \in V_0(T)$. By lemma 3, there is a normal form $\Box(a_0 \vee \cdots \vee a_n) \equiv \Box s$, then $t \models_T \Box(a_0 \vee \cdots \vee a_n)$, then we have $t' \models_{T'} \Box(a_0 \vee \cdots \vee a_n)$ since $V_0'(T) \subseteq V_0'(T')$. Therefore, $t' \models_{T'} \Box s$.

Assume that $A = \{a_1, a_2, \cdots, a_n\}$, we write $t \models_T \Box A$ standing for $t \models_T \Box(a_0 \vee \cdots \vee a_n)$.

Lemma 5. Assume that $A = \{a_0, a_1, \cdots, a_n\}$, $B = \{b_0, b_1, \cdots, b_m\}$, and $B \preccurlyeq_S A$. If $t \models_T \Box A$, then $t \models_T \Box B$.

Proof. Because that $t \models_T \Box A$, then for every maximal branch φ in the subtree out of *t*, there is a node t' which satisfies $a_0 \vee \cdots \vee a_n$, that is, $\exists a_i \in A$, *s.t.* $t' \models_T a_i$. It follows that $a_i \leq val(t')$. So there is $b_j \in B$, $b_j \leq a_i \leq val(t')$ since $B \preccurlyeq_S A$, then $t' \models_T b_j$, and $t' \models_T (b_0 \vee \cdots \vee b_m)$. By lemma 1, $\models_T \Box B$.

The following theorem show how the Smyth powerdomain is built up from assertions about the inevitable behavior of a process. Winskel gave the sketch of the proof [], but it is fuzzy to understand. Here we give a kind of direct proof. The ideal is very simple, but there is a new trick to construct a finite branching tree by an element of Smyth powerdomain.

Theorem 1. *Let* $\mathbf{T}$ *be the class of nondeterministic D-computations. The Smyth powerdomain* $\mathbf{P_0}[\mathbf{D}]$ *is isomorphic to the quotient* $(\mathbf{T}/\simeq_0, \preccurlyeq_0/\simeq_0)$, *and to the order* $(\{V_0(T) | T \in \mathbf{T}\}, \subseteq)$.

Proof. By the lemma 4, it suffices to prove that $P_0[D]$ is isomorphic to $(\{V_0'(T) | T \in \mathbf{T}\}, \subseteq)$. Define $f: (\{V_0'(T) | T \in \mathbf{T}\}, \subseteq) \longrightarrow (\mathbf{P_0}[\mathbf{D}], \subseteq)$ as follows; for any $T \in \mathbf{T}$, $f(V_0'(T)) = \{\{a_0, a_1, \cdots, a_n\} \mid \Box(a_0 \vee \cdots \vee a_n) \in V_0'(T)\} \triangleq I(T)$.

Firstly, we prove that $I(T) \in (\mathbf{P_0}[\mathbf{D}], \subseteq)$. It is clear that $\{\perp\} \in I(T)$. so $I(T) \neq \emptyset$. Assume that $B \triangleq \{b_0, b_1, \cdots, b_p\} \preccurlyeq_S A \triangleq \{a_0, a_1, \cdots, a_q\}$, and $A \in I(T)$. By the lemma 5, $t \models_T \Box B$. Then $B \in I(T)$. On the other hand, let $A = \{a_0, ..., a_n\}$, $B = \{b_0, ..., b_m\} \in I(T)$, that is, $\models_T \Box(a_0 \vee \cdots \vee a_n)$ and $\models_T \Box(b_0 \vee \cdots \vee b_m)$. Then every maximal branch φ in *T* has a node t' which satisfies $a_0 \vee \cdots \vee a_n$ and a node t'' which satisfies $b_0 \vee \cdots \vee b_m$. So, just like lemma 1, we can construct two finite subtrees T_A and T_B of T where the leaves of T_A satisfy $a_0 \vee \cdots \vee a_n$ and the leaves of T_B satisfy $b_0 \vee \cdots \vee b_m$. Let C be a set as follows; for every maximal branch of T, if the leave t_A of T_A is less than the leave t_B of T_B, then $val(t_B) \in C$, otherwise, $val(t_A) \in C$. Because T_A and T_B are finite, C is a finite set. $C \in I(T)$is clear. For any $val(t) \in C$, $t \models_T a_0 \vee \cdots \vee a_n$ and $t \models_T b_0 \vee \cdots \vee b_m$, so there exist a_i, b_j so that $t \models_T a_i$, $t \models_T b_j$, that is, $a_i \leq val(t)$ and $b_j \leq val(t)$. Hence C is a upper bound w.r.t $\preccurlyeq_S$ of A and B.

Secondly, if $V_0'(T_1) = V_0'(T_2)$, then the fact that $f(V_0'(T_1)) = f(V_0'(T_2))$ is clear.

Thirdly, the fact that f is one-one is clear.

Next, we prove f is onto.

Assume that $\mathcal{A} \in \mathbf{P_0}[\mathbf{D}]$. Since D^0 is countable, then we can know that $\mathcal{A}$ is countable and nonempty ($\{\perp\} \in \mathcal{A}$). So we can assume $\mathcal{A} = \{B_0', \cdots, B_n', \cdots\}$. Let A_0 be B_0'; let A_1 be an upper bound $w.r.t. \preccurlyeq_S$ of A_0 and B_1'; $\cdots$; let A_n be an upper bound $w.r.t. \preccurlyeq_S$ of A_{n-1} and B_n'; $\cdots$. Hence, we get a $\omega-$chain $A_0 \preccurlyeq_S A_1 \preccurlyeq_S \cdots \preccurlyeq_S A_n \preccurlyeq_S \cdots$. Define a function $\varphi_n : A_{n+1} \longrightarrow A_n$ where for any $a \in A_{n+1}$, $\varphi_n(a) \leq a$.

When $m > n$ denote $\varphi_n(\varphi_{n+1}(\cdots(\varphi_{m-1}\varphi_m)\cdots)) : A_{m+1} \longrightarrow A_n$ as φ_{nm}

Let $A_n' = \bigcap_{m>n} \varphi_{nm}(A_{m+1})$, $n = 0, 1, 2, \cdots$

We construct T as follows;

let $\perp$ be the root of *T*.

A_n' is the set of nodes of *T* at the height n+1

$a_{nk} \longrightarrow a_{n+1k'}$ iff $a_{nk} = \varphi_n(a_{n+1k'})$

So $(T, \rightarrow, id)$ is a nondeterministic D-computation

$(T, \rightarrow, id)$ has the following properties;

(1) $A_n' \preccurlyeq_S A_{n+1}'$.

For any $a_{n+1}' \in A_{n+1}'$, that is, for any $m > n+1, a_{n+1}' \in \varphi_{n+1m}(A_{m+1})$, there is an $a_{m+1} \in A_{m+1}$ such that $a_{n+1}' = \varphi_{n+1m}(a_{m+1})$. Then $\varphi_n(a_{n+1}') = \varphi_n(\varphi_{n+1m}(a_{m+1})) = \varphi_{nm}(a_{m+1}) \in \varphi_{nm}(A_{m+1}) \subseteq \varphi_n\varphi_{n+1}(A_{n+2})$. So, $\varphi_n(a_{n+1}') \in \bigcap_{m>n} \varphi_{nm}(A_{m+1})$, that is, $\varphi_n(a_{n+1}') \in A_n'$ and $\varphi_n(a_{n+1}') \leq a_{n+1}'$. Hence, $A_n' \preccurlyeq_S A_{n+1}'$.

(2) For any n, $A_n' \in \mathcal{A}$

$A_n' = \bigcap \varphi_{nm}(A_{m+1})$. If for any m, there is an $m' > m$ such that $\varphi_{nm'}(A_{m'+1}) \neq \varphi_{nm}(A_{m+1})$, that is, $\varphi_{nm'}(A_{m'+1}) \subset \varphi_{nm}(A_{m+1})$. Let $k_1 > n+1$ be a least number so that $\varphi_{nk_1}(A_{k_1+1}) \subset \varphi_{nm}(A_{m+1})$.

Let $C_1 = \varphi_{nm}(A_{m+1}) - \varphi_{nk_1}(A_{k_1+1}) \neq \emptyset$; Let $k_2 > k_1$ be a least number so that $\varphi_{nk_2}(A_{k_2+1}) \subset \varphi_{nk_1}(A_{k_1+1})$. Let $C_2 = \varphi_{nk_1}(A_{k_1+1}) - \varphi_{nk_2}(A_{k_2+1}) \neq \emptyset$;

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Let $C = \bigcup C_i$. So C is an infinite set which contradicts the fact that $C \subseteq \varphi_n(A_{n+1})$ is a finite set. Hence, there exists an m, for any $m' > m$, $\varphi_{nm'}(A_{m'+1}) = \varphi_{nm}(A_{m+1})$. Then for any n, there is an m so that $A_n' = \varphi_{nm}(A_{m+1})$. Because $\varphi_{nm}(A_{m+1}) \preccurlyeq_S A_{m+1}$, then $A_n' \in \mathcal{A}$.

(3) For any node a_{nk} of T at the height $n+1$, there is a node $a_{n+1k'}$ of T at the height $n+2$ such that $a_{nk} \longrightarrow a_{n+1k'}$.

According to property (2), there is an $m > n+1$ such that $A_n' = \varphi_{nm}(A_{m+1})$ and $A_{n+1}' = \varphi_{n+1m}(A_{m+1})$. Because $a_{nk} \in A_n'$, there is an $a_{m+1} \in A_{m+1}$ such that $a_{nk} = \varphi_{nm}(a_{m+1}) = \varphi_n(\varphi_{n+1m}(a_{m+1}))$.

Let $a_{n+1k'} = \varphi_{n+1m}(a_{m+1}) \in A_{n+1}'$. So $a_{n+1k'}$ is node in the *T* at the height n+2. We have $a_{nk} = \varphi_n(a_{n+1k'})$, that is, $a_{nk} \longrightarrow a_{n+1k'}$.

(4) For any A_n', $t \models_T \Box A_n'$

According to property (3), for any maximal branch ϕ in the subtree out of t, there is an element of A_n' is one of the node of ϕ. By the lemma 1, $t \models_T \Box A_n'$

Now we prove that $f(V_0'(T)) = \mathcal{A}$

For any $\Box(a_0 \vee \cdots \vee a_n) \in V_0'(T)$, because $\models_T \Box(a_0$

$\vee \cdots \vee a_n$), then for any maximal branch in the T, there is a node which satisfies $a_0 \vee \cdots \vee a_n$. According to the proof of the lemma 1, there is a finite subtree T' whose leaves satisfy $a_0 \vee \cdots \vee a_n$. Assume that the height of the highest leave is m. According to property (3), the set of leaves of T' is less than A'_m w.r.t. $\preccurlyeq_S$, then $\{a_0, \cdots, a_n\} \preccurlyeq_S A'_m$. From $A'_m \in \mathcal{A}$, we get $\{a_0, ..., a_n\} \in \mathcal{A}$. On the other hand, for any $A = \{a_0, ..., a_n\} \in \mathcal{A}$, by the construction of $\omega-$chain, there is a A_n such that $A \preccurlyeq_S A_n$. Since $A_n \preccurlyeq_S A'_n$, then $A \preccurlyeq_S A'_n$. From the property (4) and the lemma 4, $t \models_T \Box A$. So we can prove that $A \in f(V'_0(T))$.

Finally, the fact that $V_0{}'(T) \subseteq V_0{}'(T') \iff f(V_0{}'(T)) \subseteq f(V_0{}'(T'))$is clear.

3 Hoare Powerdomain

To get Hoare powerdomain, we look at assertions built using the logic model which standard 'possibly' operator. In fact, Hoare powerdomain has an even simpler construction.

Lemma 6. Let $\mathcal{L}(D^0)$ consist of the non-null, left-closed subsets of D^0, then $(\mathcal{L}(D^0), \subseteq)$ is isomorphic to $(P_1[D], \subseteq)$, the Hoare powerdomain.

Proof. Let F be a function from $\mathcal{L}(D^0)$ to $P_1[D]$ as follows.

Firstly, $F(X) \neq \emptyset$ since X is non-null. For any $A, B \in M[D]$, if $A \preccurlyeq_H B$ and $B \in F(X)$, then for any $a \in A$, there is a $b \in B \subseteq X$ such that $a \leq b$, so $a \in X$ by definition of X. Therefore $A \in F(X)$. For any $A, B \in F(X)$, then $A \preccurlyeq_H A \cup B$ and $B \preccurlyeq_H A \cup B$ and $A \cup B \in F(X)$. Secondly, if $X_1 = X_2$, then $F(X_1) = F(X_2)$. Therefore, F is well-defined.

Let $F^\star$ be a function from $P_1[D]$ to $\mathcal{L}(D^0)$ as follows, $F^\star(\mathcal{A}) = \cup\{A \mid A \in \mathcal{A}\}$

Firstly, $F^\star(\mathcal{A}) \in \mathcal{L}(D^0)$. In fact, since $\{\perp\} \in \mathcal{A}, F^\star(\mathcal{A}) \neq \emptyset$. For any a, b, $a \leq b \in F^\star(\mathcal{A})$, then there is an A in $\mathcal{A}$ such that $b \in A$. So $\{a\} \preccurlyeq_H \{b\} \preccurlyeq_H A$. then we have $\{a\} \in \mathcal{A}$, hence, $a \in F^\star(\mathcal{A})$. Secondly, if $\mathcal{A}_1 = \mathcal{A}_2$, then $F^\star(\mathcal{A}_1) = F^\star(\mathcal{A}_2)$. Therefore, $F^\star$ is also well-defined.

Next, we prove that the follow results.

(1) $FF^\star(\mathcal{A}) = \mathcal{A}, \forall \mathcal{A} \in P_1[D]$;

It follows that $FF^\star(\mathcal{A}) = \{X \mid X \subseteq F^\star(\mathcal{A}), X\ is\ finite\} = \{X \mid X \subseteq \cup\{A \mid A \in \mathcal{A}\}, X\ is\ finite\} = \{A \mid A \in \mathcal{A}\} = \mathcal{A}$

(2) $F^\star F(X) = X, \forall X \in \mathcal{L}(D^0)$

It follows that $F^\star F(X) = \cup\{A \mid A \in F(X)\} = \cup\{A \mid A \in \{A \mid A \subseteq X, A\ is\ finite\}\} = X$

Now, we define this logic model and the satisfaction relation.

Let $(T, \to, val)$ be a nondeterminstic D-computation. Define $\models_T$ to be the least relation included in $T \times L_1$ such that $t \models a\ \ if\ \ a \leq val(t)$, $t \models \Diamond s\ \ if\ \ t \models s\ \ or\ \ (\exists t', t' \to t \Longrightarrow t' \models \Diamond s)$.

Here we are interested in those possible statements.

Definition 3. *Let* $T = (T, \to, val)$ *be a nondeterministic* D-computation *with root node* t. *Define* $V_0(T) = \{\Diamond s \in L_1 \mid \models_T \Diamond s\}$.

Based on this assertions, we define an obvious preorder on nondeterministic computations, $T \preccurlyeq_1 T' \iff V_1(T) \subseteq V_1(T')$

Quotienting the preorder $\preccurlyeq_1$ on nondeterministic computations by the equivalence $\simeq_1 \triangleq \preccurlyeq_1 \cap \preccurlyeq_1{}^{-1}$, we obtain the Hoare powerdomain by theorem 2.

Lemma 7. For all $s \in L_1$, $\Diamond s \in L_1$ is $\equiv$-equivalent a normal form $\Diamond a$, for $a \in D^0$.

Proof. By definition of L_1, s is any formula built with the following syntax,

$$s ::= a \mid \Diamond s$$

We proceed by induction on s

(1) The basic case is that $s = a$. Then $\Diamond s = \Diamond a$, that is, $\Diamond s$ is $\equiv$-equivalent the normal form $\Diamond a$.

(2) The other case is that $s = \Diamond s'$. Suppose that $\Diamond s'$ is $\equiv$-equivalent a normal form $\Diamond a$. Then $\Diamond s \equiv \Diamond(\Diamond s') \equiv \Diamond s' \equiv \Diamond a$.

Let $V'_1(T) = \{\Diamond a \mid \models_T \Diamond a, a \in D^0\}$.

Lemma 8. Let T and T' be nondeterministic D-computations. The node t is the root of T, and the node t' is the root of T', we have $V_1(T) \subseteq V_1(T') \Leftrightarrow V'_1(T) \subseteq V'_1(T')$.

Proof. We assume that $V_1(T) \subseteq V_1(T')$. For any $\Diamond a \in V'_1(T)$, we have $\models_T \Diamond a$. Then $\Diamond a \in V_1(T) \subseteq V_1(T')$. So $\models_{T'} \Diamond a$. Therefore, $\Diamond a \in V'_1(T')$. On the other hand, we assume that $V'_1(T) \subseteq V'_1(T')$. For any $\Diamond s \in V_1(T)$, we have $\models_T \Diamond s$, then there is an $a \in D^0$ such that $\Diamond s \equiv \Diamond a$ by the lemma 7, that is, $\models_T \Diamond a$, so $\Diamond a \in V'_1(T) \subseteq V'_1(T')$. Then we have $\models_{T'} \Diamond a$. Therefore, $\models_{T'} \Diamond s$, that is, $\Diamond s \in V_1(T')$.

Theorem 2. Let **T** be the class of nondeterministic D-computation. The Hoare powerdomain $P_1[D]$ is isomorphic to the quotient $(T/\simeq_1, \preccurlyeq_1/\simeq_1)$, and to the order $(\{V_1(T) \mid T \in \mathbf{T}\}, \subseteq)$.

Proof. By lemma 6, and lemma 8, we just need to prove that $(\mathcal{L}(D^0), \subseteq)$ is isomorphic to $(\{V'_1(T) \mid T \in \mathbf{T}\}, \subseteq)$. Define $f : (\{V'_1(T) \mid T \in \mathbf{T}\}, \subseteq) \longrightarrow (\mathcal{L}(D^0), \subseteq)$

For any $T \in \mathbf{T}$, $f(V'_1(T)) = \{a \in D^0 \mid \models_T \Diamond a\} \triangleq I$. Firstly, we prove that $I \in \mathcal{L}(D^0)$. It is clear that $\perp \in I$, so $I \neq \emptyset$. Assume that $a, b \in D^0$, $a \leq b$ and $b \in I$, then we have $\models_T \Diamond b$, that is, there is a finite branch from the root t to t' which t' satisfies b. So $b \leq val(t')$. Since $a \leq b \leq val(t')$. t' also satisfies a. Hence $\models_T \Diamond a$, $a \in I$. Secondly, if $V'_1(T) = V'_1(T')$, then $f(V'_1(T)) = f(V'_1(T'))$.

Thirdly, the fact that f is one-one is clear.

Next we prove that f is onto.

Assume that $X \in \mathcal{L}(D^0)$. Since $\perp \in X$, we assume that $X = \{\perp, a_0, a_1, \cdots, a_n, \cdots\}$ We construct T as follows;

The fact that $f(V'_1(T)) = X$ is clear.

Finally, the fact that $V_1{}'(T) \subseteq V_1{}'(T') \iff f(V_1{}'(T)) \subseteq f(V_1{}'(T'))$ is clear.

4 Plotkin Powerdomain

From the two sections above, we can find the same way to obtain the plotkin powerdomain. In fact, it is obtained by considering information about both the inevitable and possible behaviour of a computation.

We define a logic modal L_2 including D^0, for all $s \in L_2$, s is any formula built with the following syntax;

$$s ::= a \mid \Box s \mid s \vee s \vee \Diamond s \ (a \in D^0)$$

Let $(T, \to, val)$ be a nondeterminstic D-computation. Define $\models_T$ to be the least relation included in $T \times L_2$ such that

$$t \models a \quad if \quad a \leq val(t)$$
$$t \models (s \vee s') \quad if \quad t \models s \quad or \quad t \models s'$$
$$t \models \Box s \quad if \quad t \models s \quad or \quad (\forall t', t' \to t \Longrightarrow t' \models \Box s)$$
$$t \models \Diamond s \quad if \quad t \models s \quad or \quad (\exists t', t' \to t \Longrightarrow t' \models \Diamond s)$$

In our proofs, the following properties are also needed.

Lemma 9. *$t \models_T \Diamond s$ if and only if there is a finite branch from t to t' with the node t' satisfying s.*

Proof. Suppose that $t \models_T \Diamond s$. To prove it, by the definition of $\models_T$, we are only allowed to use the following rules.

The basic case is that from $t \models_T s$ we can get $t \models_T \Diamond s$. Then of course, there is a finite branch from t to t and the node t satisfies s. The other case is that if there is a t' such that $t \to t'$ and we have $t' \models_T \Diamond s$, then we can get $t \models_T \Diamond s$. By induction we may assume that for t', there is a finite branch ϕ from t' to t'' with the node t'' satisfying s. Let ϕ' be a branch which $t \to t'$ has been added to ϕ. Then ϕ' is a finite branch from t to t'' with the node t'' satisfying s. On the other hand, suppose that there is a finite branch ϕ from t to t' with the node t' satisfying s. Let the height of the branch ϕ be n. For any node t_h of the branch ϕ, let h be the height from t_h to t' in ϕ. When $h = 0$, $t_h = t'$, then from $t' \models_T s$, we have $t_h \models_T \Diamond s$ by rule (1). Assume when $h < n$, $t_h \models_T \Diamond s$, then when $h = n$, we have $t_n \models_T \Diamond s$ by the a rule (2). Hence, $t \models_T \Diamond s$.

Lemma 10. Let $T = (T, \to, val)$ be a nondeterministic D-computation with root node t. Write $\models_T s$ for $t \models_T s$. Let $\mathbf{T}$ be the class of nondeterministic D-computations. Define $s \equiv s' \ iff \ (\models_T s \Leftrightarrow \models_T s', \forall T \in \mathbf{T})$.

We have

(1) $s \equiv s'$ iff $(\models_T s \Leftrightarrow \models_T s', \forall T \in \mathbf{T})$;

(2) $s \vee s' \equiv s' \vee s$;

(3) $\Box(\Box s) \equiv \Box s$;

(4) $\Box(s \vee \Box s') \equiv \Box(s \vee s')$;

(5) $\Diamond(s \vee s') \equiv \Diamond s \vee \Diamond s'$;

(6) $\Diamond(\Diamond s) \equiv \Diamond s \equiv \Box(\Diamond s) \equiv \Diamond(\Box s)$;

(7) $\Box(s \vee (\Diamond s')) \equiv (\Box s) \vee (\Diamond s')$;

(8) $s \equiv s' \Longrightarrow \Box s \equiv \Box s'$;

(9) $s \equiv s' \Longrightarrow \Diamond s \equiv \Diamond s'$;

(10) $s \equiv s' \Longrightarrow s \vee s'' \equiv s' \vee s''$

Proof. The proofs of (1), (2), (3), (4), (8), (10) are seen at lemma 2.

(5) For any $T \in \mathbf{T}$, if $t \models_T \Diamond(s \vee s')$, then there is finite branch from t to t' with the node t' satisfying $s \vee s'$. If t' satisfies s, then $t \models_T \Diamond s$, of course, $t \models_T \Diamond s \vee \Diamond s'$. The case is similar if t' satisfies s'. On the other hand, if $t \models_T \Diamond s \vee \Diamond s'$, then $t \models_T \Diamond s$ or $t \models_T \Diamond s'$. If $t \models_T \Diamond s$, then there is a finite branch from t to t' with the node t' satisfying s, so $t' \models_T s \vee s'$. The case is similar if $t \models_T \Diamond s'$. Hence, $t \models_T \Diamond(s \vee s')$.

(6) For any $T \in \mathbf{T}$, if $t \models_T \Diamond(\Diamond s)$, then there is a finite branch ϕ from t to t' with the node t' satisfying $\Diamond s$. So there is another finite branch ϕ' from t' to t'' with the node t'' satisfying s. Define that $\phi + \phi'$ as ϕ to which ϕ' has been added. It is clear that $\phi + \phi'$ is a finite branch from t to t'' with the node t'' satisfying s. Then $t \models_T \Diamond s$.

For any $T \in \mathbf{T}$, if $t \models_T \Diamond s$, then $t \models_T \Box(\Diamond s)$.

For any $T \in \mathbf{T}$, if $t \models_T \Box(\Diamond s)$, then $t' \models_T \Box s$. Hence, $t \models_T \Diamond(\Box s)$.

For any $T \in \mathbf{T}$, if $t \models_T \Diamond(\Box s)$, then there is a finite branch from t to t' with the node t' satisfying $\Box s$. So any maximal branch in the subtree out of t' has a node which satisfies s, of course, there is a finite branch from t' with a node satisfying s, that is, $t' \models_T \Diamond s$. Hence, $t \models_T \Diamond(\Diamond s)$.

So far, we have $\Diamond(\Diamond s) \equiv \Diamond s \equiv \Diamond(\Box s)$.

(7) For any $T \in \mathbf{T}$, if $t \models_T \Box(s \vee (\Diamond s'))$, then for any maximal branch ϕ in the subtree out of t there is a node t_ϕ which satisfies $s \vee (\Diamond s')$. If there is a maximal branch ϕ such that $t_\phi \models_T \Diamond s'$, then there is a finite branch from t_ϕ to t' with the t' satisfying s. Hence, there is a finite branch from t to t' via t_ϕ with t' satisfying s, namely, $t \models_T \Diamond s$, of course, $t \models_T (\Box s) \vee (\Diamond s')$. If for any maximal branch ϕ, $t_\phi \models_T s$, then $t \models_T \Box s$, of course, $t \models_T (\Box s) \vee (\Diamond s')$.

On the other hand, if $t \models_T (\Box s) \vee (\Diamond s')$, then $t \models_T \Box s$ or $t \models_T \Diamond s'$. If $t \models_T \Box s$, then any maximal branch in the subtree out of t has a node which satisfies s, of coure satisfies $s \vee (\Diamond s')$. Hence, $t \models_T \Box(s \vee (\Diamond s'))$. If $t \models_T \Diamond s'$, then $t \models_T s \vee (\Diamond s')$, hence, $t \models_T \Box(s \vee (\Diamond s'))$.

(9) It is clear.

In the logic model L_2, the normal forms are following;

Lemma 11. *For all $s \in L_2$, $\Box s \in L_2$ is $\equiv -$ equivalent a normal form $\Box(a_0 \vee \cdots \vee a_n) \vee \Diamond b_0 \vee \cdots \vee \Diamond b_m$ for some $a_0, \cdots, a_n, b_0, \cdots, b_m \in D^0$.*

Proof. By definition of L_2, s is any formula built with the following syntax, $s ::= a | \Box s | \Diamond s | s \vee s$.

We proceed by induction on s

(1) The basic case is that $s = a$. Then $\Box s = \Box a$, that is, $\Box s$ is $\equiv -$ equivalent the normal form $\Box a$.

(2) Another case is that $s = \Box s'$. Assume that $\Box s'$ is $\equiv -$ equivalent a normal form $\Box(a_0 \vee \cdots \vee a_n) \vee \Diamond b_0 \vee \cdots \vee \Diamond b_m$. Then $\Box s = \Box(\Box s') \equiv \Box s' \equiv \Box(a_0 \vee \cdots \vee a_n) \vee \Diamond b_0 \vee \cdots \vee \Diamond b_m$ by the lemma 10.

(3) Another case is that $s = s' \vee s''$. Suppose that $\Box s'$ is $\equiv -$ equivalent a normal form $\Box(a_0 \vee \cdots \vee a_n) \vee \Diamond b_0 \vee \cdots \vee \Diamond b_m$ and s'' is $\equiv -$ equivalent a normal form $\Box(c_0 \vee \cdots \vee c_p) \vee \Diamond d_0 \vee \cdots \vee \Diamond d_q$. Then $\Box s = \Box(s' \vee s'') \equiv \Box(s' \vee \Box s'') \equiv \Box(\Box s'' \vee s') \equiv \Box(\Box s'' \vee \Box s') \equiv \Box(\Box(c_0 \vee \cdots \vee c_p) \vee \Diamond d_0 \vee \cdots \vee \Diamond d_q \vee \Box(a_0 \vee \cdots \vee a_n) \vee \Diamond b_0 \vee \cdots \vee \Diamond b_m) \equiv \Box(\Box(c_0 \vee \cdots \vee c_p) \vee \Box(a_0 \vee \cdots \vee a_n) \vee \Diamond d_0 \vee \cdots \vee \Diamond d_q \vee \Diamond b_0 \vee \cdots \vee \Diamond b_m) \equiv \Box(\Box(c_0 \vee \cdots \vee c_p) \vee \Box(a_0 \vee \cdots \vee a_n) \vee \Diamond d_0 \vee \cdots \vee \Diamond d_q \vee \Diamond b_0 \vee \cdots \vee \Diamond b_m) \vee \Box(a_0 \vee \cdots \vee a_n)) \vee \Diamond d_0 \vee \cdots \vee \Diamond d_q \vee \Diamond b_0 \vee \cdots \vee \Diamond b_m \equiv \Box(\Box(c_0 \vee \cdots \vee c_p) \vee (a_0 \vee \cdots \vee a_n)) \vee \Diamond d_0 \vee \cdots \vee \Diamond d_q \vee \Diamond b_0 \vee \cdots \vee \Diamond b_m \equiv \Box(a_0 \vee \cdots \vee a_n \vee$

$\Box(c_0 \vee \cdots \vee c_p)) \vee \Diamond d_0 \vee \cdots \vee \Diamond d_q \vee \Diamond b_0 \vee \cdots \vee \Diamond b_m \equiv \Box(a_0 \vee \cdots \vee a_n \vee c_0 \vee \cdots \vee c_p) \vee \Diamond d_0 \vee \cdots \vee \Diamond d_q \vee \Diamond b_0 \vee \cdots \vee \Diamond b_m$ by the lemma 10.

(4) Another case is that $s = \Diamond s'$. Suppose that s' is $\equiv$-equivalent a normal form $\Box(a_0 \vee \cdots \vee a_n) \vee \Diamond b_0 \vee \cdots \vee \Diamond b_m$. Then $\Box s = \Box(\Diamond s') \equiv \Diamond(\Box s') \equiv \Diamond(\Box(a_0 \vee \cdots \vee a_n) \vee \Diamond b_0 \vee \cdots \vee \Diamond b_m) \equiv \Diamond(\Box(a_0 \vee \cdots \vee a_n)) \vee \Diamond(\Diamond b_0 \vee \cdots \vee \Diamond b_m) \equiv \Diamond(a_0 \vee \cdots \vee a_n) \vee \Diamond b_0 \vee \cdots \vee \Diamond b_m \equiv \Diamond a_0 \vee \cdots \vee \Diamond a_n \vee \Diamond b_0 \vee \cdots \Diamond b_m$ by the lemma 10.

The same as Smyth powerdomain, here we are only interested in that information $\Box((\Diamond a) \vee b)$.

Definition 4. Let $T = (T, \rightarrow, val)$ be a nondeterministic D-computation with root node t. Define $V_2(T) = \{\Box s \in L_2 \mid \models_T \Box s\}$

Based on this assertions, we define an obvious preorder on nondeterministic computations, $T \preccurlyeq_2 T' \iff V_2(T) \subseteq V_2(T')$

Quotienting the preorder $\preccurlyeq_2$ on nondeterministic computations by the equivalence $\simeq_2 \triangleq \preccurlyeq_2 \cap \preccurlyeq_2{}^{-1}$, we obtain the Plotkin powerdomain in theorem 3.

Let $V_2'(T) = \{\Box(a_0 \vee a_1 \vee \cdots \vee a_n\} \mid \models_T \Box(a_0 \vee \cdots \vee a_n), t \models_T \Diamond a_i, i = 0, ..., n\}$.

Lemma 12. Let T and T' be nondeterministic D-computations. The node t is the root of T, and the node t' is the root of T'. We have $V_2(T) \subseteq V_2(T') \Leftrightarrow V_2'(T) \subseteq V_2'(T')$

Proof. We assume that $V_2(T) \subseteq V_2(T')$. For any $\Box(a_0 \vee \cdots \vee a_n) \in V_2'(T)$, we have $t \models_T \Box(a_0 \vee \cdots \vee a_n)$, and $t \models_T \Box(\Diamond a_i)$ (Since $\Box(\Diamond a_i) \equiv \Diamond a_i$), $i = 0, \cdots, n$. Then we have $\Box(a_0 \vee \cdots \vee a_n) \in V_2(T)$ and $\Box(\Diamond a_i) \in V_2(T)$. Since $V_2(T) \subseteq V_2(T')$, we have $t' \models_{T'} \Box(a_0 \vee \cdots \vee a_n)$ and $t' \models_{T'} \Box(\Diamond a_i)$, $i = 0, \cdots, n$. Then $t' \models_{T'} \Diamond a_i$ (Also since $\Diamond a_i \equiv \Box(\Diamond a_i)$), $i = 0, \cdots, n$. So $\Box(a_0 \vee \cdots \vee a_n) \in V_2'(T')$.

On the other hand, we assume that $V_2'(T) \subseteq V_2'(T')$. Let $\Box s \in V_2(T)$, we have $t \models_T \Box s$. By the lemma 11, $\Box s \equiv \Box(a_0 \vee \cdots \vee a_k) \vee \Diamond a_{k+1} \vee \cdots \vee \Diamond a_n$, where $a_0, \cdots, a_n \in D^0$. So $t \models_T \Box(a_0 \vee \cdots \vee a_k) \vee \Diamond a_{k+1} \vee \cdots \vee \Diamond a_n$. If $t \models_T \Box(a_0 \vee \cdots \vee a_k)$, there is a finite subtree T' where the leaves satisfy $a_0 \vee \cdots \vee a_k$. Let $\{b_1, b_2, \cdots, b_l\} \subseteq \{a_0, \cdots, a_k\}$, where for any $b_j, j = 1, \cdots, l$, there is some leave of T' satisfy b_j. Then $t \models_t \Box(b_1 \vee \cdots \vee b_l)$ and $t \models_T \Diamond b_j, j = 1, \cdots, b_l$. So $\Box(b_1 \vee b_2 \vee \cdots \vee b_l) \in V_2'(T)$, Since $V_2'(T) \subseteq V_2'(T')$, $t' \models_{T'} \Box(b_1 \vee b_2 \vee \cdots \vee b_l)$ and $t' \models_{T'} \Diamond b_j, j = i, \cdots, l$, of course, $t' \models_{T'} \Box(a_0 \vee \cdots \vee a_k) \vee \Diamond a_{k+1} \vee \cdots \vee \Diamond a_n$.

If $t \models_T \Diamond a_i$, there is a finite branch from t to s_j with the node s_j satisfying a_i. Let h be the height from t to s_j. Let T' be a subtree of T where the nodes of T' are the nodes of T which height is less than and equal to h. Assume that the set of leaves of T' is $\{s_1, s_2, \cdots, s_j, \cdots, s_n\}$, then it is easy to prove that $t \models_T \Box(val(s_1) \vee \cdots val(s_j) \vee \cdots \vee val(s_n))$, and $t \models_T \Diamond val(s_i), i = 1, \cdots, n$, So $\Box(val(s_1) \vee \cdots \vee val(s_n)) \in V_2'(T)$, since $V_2'(T) \subseteq V_2'(T')$, $t' \models_{T'} \Box(val(s_1) \vee \cdots val(s_j) \vee \cdots \vee val(s_n))$ and $t' \models_{T'} \Diamond val(s_i), i = 1, \cdots, n$. Specially, $t' \models_{T'} \Diamond val(s_j)$, So, $t' \models_{T'} \Diamond a_i$. Therefore, $t' \models_{T'} \Box(a_0 \vee \cdots \vee a_k) \vee \Diamond a_{k+1} \vee \cdots \vee \Diamond a_n$. Hence $V_2(T) \subseteq V_2(T')$.

Lemma 13. Let $B = \{b_0, ..., b_m\}$, $A = \{a_0, ..., a_n\}$. $B \preccurlyeq_P A$. If $t \models_T \Box A$ and $t \models_T \Diamond a_i$, for any $a_i \in A$, then $t \models_T \Box B$ and $t \models_T \Diamond b_j$, for any $b_j \in B$.

Proof. Because $t \models_T \Box(a_0 \vee \cdots \vee a_n)$, and $t \models_T \Diamond a_i, i = 0, ..., n$. So for every maximal branch ϕ in the subtree out of t, there is a node t' which satisfies $a_0 \vee \cdots \vee a_n$, that is, there is a $a_i \in A$ such that $t' \models_T a_i$. It follows $a_i \leq val(t')$. Because $B \preccurlyeq_S A$, then there is a $b_j \in B$ such that $b_j \leq a_i \leq val(t')$, that is, $t' \models_T b_j$. Therefore $t \models_T \Box(b_0 \vee \cdots \vee b_m)$. On the other hand, for any $b_j \in B$, because $B \preccurlyeq_H A$, there is $a_i \in A$ such that $b_j \leq a_i$. Because $t \models_T \Diamond a_i$, there is a finite branch from t to t'' where t'' satisfies a_i, that is, $a_i \leq val(t'')$. Hence, $b_j \leq a_i \leq val(t'')$, that is, $t'' \models_T b_j$. So we have $t \models_T \Diamond b_j, j = 0, .., m$.

Theorem 3. Let $\mathbf{T}$ be the class of nondeterministic D-computations. The Plotkin powerdomain $\mathbf{P_2}[\mathbf{D}]$ is isomorphic to the quotient $(\mathbf{T}/\simeq_2, \preccurlyeq_2/\simeq_2)$ and to the order $(\{V_2(T) | T \in \mathbf{T}\}, \subseteq)$.

Proof. By lemma 12, it suffices to prove that $P_2[D]$ is isomorphic to $(\{V_2'(T) | T \in \mathbf{T}\}, \subseteq)$ Define $f : (\{V_2'(T) | T \in \mathbf{T}\}, \subseteq) \longrightarrow (\mathbf{P_2}[\mathbf{D}], \subseteq)$ as follows;

for any $T \in \mathbf{T}$, $f(V_2'(T)) = \{\{a_0, a_1, \cdots, a_n\} \mid t \models_T \Box(a_0 \vee \cdots \vee a_n), t \models_T \Diamond a_i, i = 0, ..., n\} \triangleq I(T)$.

Firstly, we prove that $I(T) \in (\mathbf{P_2}[\mathbf{D}], \subseteq)$.

It is clear that $\{\perp\} \in I(T)$, so $I(T) \neq \emptyset$. Assume that $B \triangleq \{b_0, \cdots, b_m\} \preccurlyeq_P A \triangleq \{a_0, \cdots, a_n\}$, and $A \in I(T)$. Since $A \in I(T)$, from the lemma 13, we have $B \in I(T)$. Assume that $A \triangleq \{a_0, \cdots, a_n\}, B \triangleq \{b_0, \cdots, b_m\}$ and $A, B \in I(T)$. So from $t \models_T \Box(a_0 \vee \cdots \vee a_n)$ and $t \models_T \Box(b_0 \vee \cdots \vee b_m)$, we can construct two finite subtrees T_A and T_B of T where the leaves of T_A satisfy $a_0 \vee \cdots \vee a_n$ and the leaves of T_B satisfy $b_0 \vee \cdots b_m$. From $t \models_T \Diamond a_i$, and $t \models_T \Diamond b_j$ $(i = 0, ..., n, j = 0, ..., m)$, we have at most $n + m$ finite branches where each leave satisfies some a_i or some b_j respectively. Let C be a set of $val(t)$ where, for every branch of T, t is the highest node of the leave t_A of T_A, the leave t_B of T_B, the leave which satisfies some a_i(if it exists), and the leave which satisfies some b_j (if it exists). Because T_A and T_B are finite, then C is a finite set, and $C \in I(T)$ is clear. For any $val(t_c) \in C$, there are some a_i, b_j such that $t_c \models_T a_i$, $t_c \models_T b_j$, that is, $a_i \leq val(t_c)$and $b_j \leq val(t_c)$. Hence, $A \preccurlyeq_S C, B \preccurlyeq_S C$. On the other hand, for any $a_i, i = 0, ..., n$, because $t \models_T \Diamond a_i$, there is a finite branch from t to t_{a_i} where $t_{a_i} \models_T a_i$, from the definition of C, there is a $val(t) \in C$ such that $val(t_{a_i}) \leq val(t_c)$, so $a_i \leq val(t_{a_i}) \leq val(t_c)$. Hence, $A \preccurlyeq_H C$. Similarly, we can prove that $B \preccurlyeq_H C$. Therefore,

C is an upper bound of A and B $w.r.t \preccurlyeq_P$.

Secondly, if $V_2'(T_1) = V_2'(T_2)$, then $f(V_2'(T_1)) = f(V_2'(T_2))$.

Thirdly, the fact that f is one-one is clear.

Next, we prove that f is onto.

Assume that $\mathcal{A} \in P_2[D]$, since D^0 is countable, A is countable and nonempty($\{\perp\} \in \mathcal{A}$). So we can assume that $\mathcal{A} = \{B_0, B_1, \cdots, B_n, ...\}$. Let A_1 be B_0; let A_2 be an upper bound of A_1 and B_1 $w.r.t. \preccurlyeq_P$; $\cdots$; let A_n be an upper bound of A_{n-1} and B_{n-1} $w.r.t \preccurlyeq_P, \cdots$. Hence, we get a ω-chain $A_1 \preccurlyeq_P A_2 \preccurlyeq_P \cdots \preccurlyeq_P A_n \preccurlyeq_P \cdots$.

Let $A_0' = A_0$; Let $A_1' = A_0 \cup A_1; \cdots$;

Let $A_1' = A_0 \cup A_1; \cdots$

The set $\{A_0', A_1', ..., A_n', ...\}$ has the following properties;

(1) $A_n' \preccurlyeq_P A_{n+1}'$.

Proof. $A_{n+1}' = A_n' \cup A_{n+1}$. We have $A_n' \preccurlyeq_H A_{n+1}'$ since $A_n' \subseteq A_{n+1}'$. For any $a_{n+1}' \in A_{n+1}'$, that is, $a_{n+1}' \in A_n'$ or $a_{n+1}' \in A_{n+1}$. If $a_{n+1}' \in A_n'$, then there is the $a_{n+1}' \in A_n'$ such that $a_{n+1}' \leq a_{n+1}'$; if $a_{n+1}' \in A_{n+1}$, since $A_n \preccurlyeq_P A_{n+1}$, there is an $a_n \in A_n \subseteq A_n'$ such that $a_n \leq a_{n+1}'$. So $A_n' \preccurlyeq_S A_{n+1}'$. Hence, $A_n' \preccurlyeq_P A_{n+1}'$.

$A_n' \preccurlyeq_P A_n$

Proof. We show this claim by induction. When $n = 0$, $A_0' = A_0$ then the fact that $A_0 \preccurlyeq_P A_0$ is clear. Suppose that when $k < n$, $A_k' \preccurlyeq_P A_k$. Then when $k = n$, $A_n' = A_{n-1}' \cup A_n$. We have $A_n' \preccurlyeq_S A_n$ since $A_n \subseteq A_n'$. On the other hand, let $a_n' \in A_n'$, that is, $a_n' \in A_{n-1}'$or $a_n' \in A_n$. If $a_n' \in A_{n-1}'$, since $A_{n-1}' \preccurlyeq_P A_{n-1}$, $A_{n-1} \preccurlyeq_P A_n$, then there is an $a_n \in A_n$ such that $a_n' \leq a_n$; if $a_n' \in A_n$, then of course, there is the $a_n' \in A_n$ such that $a_n' \leq a_n'$. So $A_n' \preccurlyeq_H A_n$. Hence, $A_n' \preccurlyeq_P A_n$.

$A_n \preccurlyeq_P A_{n+1}'$.

Proof. We show this claim by induction. When $n = 0$, the fact that $A_0 \preccurlyeq_H A_1'$ is clear since $A_0 \subseteq A_0 \cup A_1 = A_1'$. Let $a_1' \in A_1'$, that is, $a_1' \in A_0$ or $a_1' \in A_1$. If $a_1' \in A_0$, there is the $a_1' \in A_0$ such that $a_1' \leq a_1'$; if $a_1' \in A_1$, since $A_0 \preccurlyeq_P A_1$, then there is an $a_0 \in A_0$ such that $a_0 \leq a_1'$. So $A_0 \preccurlyeq_S A_1'$. Hence, $A_0 \preccurlyeq_P A_1'$. Suppose that when $k < n$, $A_k \preccurlyeq_P A_{k+1}'$. Then when $k = n$, $A_{n+1}' = A_n' \cup A_{n+1}$. We have $A_n \preccurlyeq_H A_{n+1}'$ since $A_n \subseteq A_n' \subseteq A_{n+1}'$. Let $a_{n+1}' \in A_{n+1}'$, that is, $a_{n+1}' \in A_n'$ or $a_{n+1}' \in A_{n+1}$. If $a_{n+1}' \in A_n'$, since $A_{n-1} \preccurlyeq_P A_n'$, then there is an $a_{n-1} \in A_{n-1} \subseteq A_{n-1}' \subseteq A_n$ such that $a_{n-1} \leq a_{n+1}'$; if $a_{n+1}' \in A_{n+1}$, since $A_n \preccurlyeq_P A_{n+1}$, then there is an $a_n \in A_n$ such that $a_n \leq a_{n+1}'$. So $A_n \preccurlyeq_S A_{n+1}'$. Hence, $A_n \preccurlyeq_P A_n'$.

If $A_{n+1}' - A_n' \neq \emptyset$, then $A_n' \preccurlyeq_S A_{n+1}' \preccurlyeq_S A_{n+1}' - A_n'$ since $A_{n+1}' - A_n' \subseteq A_{n+1}'$.

Define a function $\varphi_n : A_{n+1}'$-$A_n' \longrightarrow A_n'$, where for any $a \in A_{n+1}'$-A_n', $\varphi_n(a) \leq a$.

We construct T as follows;

Let $\perp$ be the root of T;

A_n' is the set of nodes of T at the height $n+1$;

$\perp \longrightarrow a_{0i}$, for any $a_{0i} \in A_0'$;

$a_{nk} \longrightarrow a_{n+1k'}$ iff $a_{nk} = \varphi(a_{n+1k'})$ or $a_{nk} = a_{n+1k'}$.

So $(T, \rightarrow, id)$ is a nondeterministic D-computation.

It is easy to check that $(T, \rightarrow, id)$ has the following properties:

(i) For any *n*, $A_n' \in \mathcal{A}$.

Since $A_n' \preccurlyeq_P A_n$, and $A_n \in \mathcal{A}$, then $A_n' \in \mathcal{A}$.

(ii) For any node a_{nk} of T at the height $n+1$, there is a node $a_{n+1k'}$ of T at the height $n+2$ such that $a_{nk} \longrightarrow a_{n+1k'}$.

Because $A_{n+1}' = A_n' \cup A_{n+1}$, that is, for any $a_{nk} \in A_n'$, $a_{nk} \in A_{n+1}'$. By the construction of, $a_{nk} \longrightarrow a_{nk}$.

(iii) For any n, $\Box A_n' \in V_2'(T)$.

According to the property (ii), for any maximal branch ϕ in the subtree out of$t$, there is an element of A_n' is one of the node of ϕ. Then $t \models_T \Box A_n'$. $t \models_T \Diamond a_i$ is clear, for any $a_i \in A_n'$. Hence $A_n' \in V_2'(T)$.

Next we prove that $f(V_2'(T)) = \mathcal{A}$.

Let $\Box(a_0 \vee \cdots \vee a_n) \in V_2'(T)$, then $t \models_T \Box(a_0 \vee \cdots \vee a_n)$, and $t \models_T \Diamond a_i$, $i = 0, ..., n$, then there is a finite subtree T' whose leaves satisfy $a_0 \vee \cdots \vee a_n$. And there is at least a leave of T' which satisfies a_i, $i = 0, ..., n$. Assume that the height of the highest leave of T' is m. According to property (ii), the set of leaves of T' is less than A_m' w.r.t. $\preccurlyeq_P$, then $\{a_0, \cdots, a_n\} \preccurlyeq_P A_m'$. From $A_m' \in \mathcal{A}$, we have $\{a_0, \cdots, a_n\} \in \mathcal{A}$. On the other hand, for any $A = \{a_0, \cdots, a_n\} \in \mathcal{A}$, by the construction of ω-chain, there is an A_n such that $A \preccurlyeq_2 A_n$. From the property (3), we have $A_n \preccurlyeq_2 A_{n+1}'$, so, by the lemma 13, $A \in f(V_2'(T))$ since $\Box A_{n+1}' \in V_2'(T)$. Finally, the fact that $V_2'(T) \subseteq V_2'(T') \Leftrightarrow f(V_2'(T)) \subseteq f(V_2'(T'))$ is clear.

5 Conclusions

We give direct detailed proofs for the connection between powerdomains and logic models which can be made about nondeterministic computations. We believe there must be the other proofs. The cause that we chose this kind of proofs is that the ideals of proofs are simple but clear. In the proceeding of proofs, we prove some algebraic properties of them at the same time. Meanwhile, we take up some trick for constructing the finite branching tree, which can also be used into the other areas.

References

[1] Winskel G 1985 On powerdomains and modality *Theoretical Computer Science* **36**(36) 127-37

[2] Hennessy M C B, Plotkin G D 1979 *Full Abstraction for a Simple Parallel Programming Language. Mathematical Foundations of Computer Science* Springer Berlin Heidelberg

[3] Huet G 1993 An analysis of Böhm's theorem *Theoretical Computer*

Science **121**(1–2) 145-67
[4] Gunter C A 1990 Relating Total and Partial Correctness Interpretations of Non-Deterministic Programs *Proceedings of the 17th ACM SIGPLAN-SIGACT symposium on Principles of programming languages* 306-19 ACM
[5] Amadio R M, Curien P L 1998 *Domains and lambda-calculi* **46** Cambridge University Press

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