On tracking ability analysis of linear extended state observer for uncertain system

Shangyao Shi*, Shiping Zhao, Jun Li

Department of Measurement and Control, Sichuan University, Chengdu 610041, P. R. China

Received 1 March 2014, www.tsi.lv

Abstract

It is known that the linear active disturbance rejection control (LADRC) is very an effective approach to control the uncertain systems. The linear extended state observer (LESO) is the major link of the LADRC, so this paper presents a modified LESO, which is used to track the state variables and estimate the unknown total disturbance. Furthermore, this paper redefines the "time-scaling" of the plant, which is a function with respect to the amplitude of the unknown total disturbance. It is first time to present the specified formula of the maximum sampling-period of LESO for some existing plants. On the anther hand, the tracking ability of the designed LESO is quantitatively described in this paper. The discussions and analysis, especially the quantitative formulas presented in this paper, will help the scholars and engineers to design the LESO in practice.

Keywords: LADRC, LESO, Modified LESO, "Time-Scaling", Sampling-Period

1 Introduction

The linear extended state observer (LESO) [1], as a mayor link of linear active disturbance rejection control (LADRC), was proposed by Prof. Gao in 2003. Since then, it is applied in many industrial fields, and shows promising control performance in practice. Recently, some breakthroughs of theoretical researches on LESO were obtained, such as in this paper [2], the rigorous proof of asymptotical convergence of LESO was given with some boundary constraint. And it was proven in [3] that the convergence of non-linear extended state observer (ESO) for a class of multi-input multi-output nonlinear systems with uncertainty can be feasible if the non-linear functions for observer are properly constructed. The researches in [4] showed that estimation and tracking errors of LESO are bounded, with their bounds monotonously decreasing with their respective bandwidths for large dynamic uncertainties. In particular, the observer bandwidth and closed-loop bandwidth were presented in [1], which is helpful in understanding the LESO easy from a physical perspective, and contributes to spreading out LESO applied in various fields.

In contrast to both theoretical and qualitative analysis of the LESO, this paper makes an offer to quantitatively analyses the latent tracking ability of LESO, which decided the disturbance rejection ability of the specific controller based on LADRC. This paper has presented a modified LESO, which not only track the state variables of plant, but also estimate the unknown total disturbance including the external disturbance and unknown model disturbance but excluding the known partial model, and redefined the "time-scaling" of the plant. Moreover, it is first time to present some specified formulas to judge the tracking ability of the LESO, and present some criterion to design the LESO for some specified plants. Following the principia, described in this paper, the LESO will be properly designed soon and implement well, which is proven by the simulations in section III.

2 Analysis of the Tracking ability of the LESO

In general case, considering a nth order system, which may be nonlinear and time-varying, with a uncertain disturbance as follow:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_n = f(X, w, t) + bu \\ y = x_1 \end{cases}, \tag{1}$$

where X is the system state variable vector $[x_1 x_2 \cdots x_n]^T$, u is the control output of the controller, and y is the measure output. $f(\cdot)$ is a uncertain generalized disturbance including the unknown system dynamics module, also named total distance including the internal disturbance and the external disturbance.

As we know that the corresponding linear extended state observer (LESO) of above system (1) with differential form is constructed as follow:

* Corresponding author- Tel: +86-28-85405310; fax: +86-28-85461088;
E-mail: peter.shi@126.com

$$\begin{cases} e = y - z_1 \\ \dot{z}_1 = z_2 + \beta_1 e \\ \dot{z}_2 = z_3 + \beta_2 e \\ \vdots \\ \dot{z}_n = z_{n+1} + \beta_n e + bu \\ \dot{z}_{n+1} = \beta_{n+1} e \\ y = x_1 \end{cases}, \tag{2}$$

where the new variables z_1, z_2, …, and z_n in (2) are the estimates of state variables x_1, x_2 ,… and x_n respectively, while the variable z_{n+1}, namely known as an augmented state variable, is the estimate of $f(\cdot)$ named the total disturbance.

A problem is proposed that how to enhance the tracking ability of the specified LESO, in other words, how much little tracking errors of the state variables can be actively got by the LESO within the constraints of the sampling-period in practice. This is very important to properly design the ADRC, because it determines the disturbance rejection ability of ADRC based on LESO.

In this section, the state variables tracking ability of LESO is discussed. Let us investigate the following plant system as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_n = f(X,w,t) + g(X,t) + bu \\ y = x_1 \end{cases}, \tag{3}$$

where $f(X,w,t)$, the unknown total disturbance, including the external disturbance and the unknown part of the plant's model, $g(X,t)$ is the known part of the plant model. A modified LESO of the system (3) is proposed and constructed as follow:

$$\begin{cases} e = y - z_1 \\ \dot{z}_1 = z_2 + \beta_1 e \\ \dot{z}_2 = z_3 + \beta_2 e \\ \vdots \\ \dot{z}_n = z_{n+1} + \beta_n e + g(X,t) + bu \\ \dot{z}_{n+1} = \beta_{n+1} e \\ y = x_1 \end{cases}, \tag{4}$$

where z_{n+1}, extended state variable, estimates the unknown total disturbance, which including the unknown part of the plant's model. Owing to including the known part of the plant model in nth equation of expression (4), the modified LESO reduces the requirements of work speed of hardware, and improve the control precision and performance of the controller.

The conception of "time-scaling", denoted as ρ, is proposed and defined in [5]. Similarly, there the "time-scaling" of the plant (3) can be defined as bellow.

Definition 2.1: the plant's "time-scaling", denoted as ρ, of plant (3) is redefined as

$$\rho = \frac{1}{M^{\frac{1}{n}}}, \tag{5}$$

where $M = \max\limits_{\substack{|w|<r_0 \\ |x_1|<r_1 \\ |x_2|<r_2 \\ \vdots \\ |x_n|<r_n}} f(x_1, x_2, \ldots, x_n, w, t)$, and $r_0, r_1, \cdots, r_n$ are bounded for any $0 \le t < \infty$. And the plant with $\rho = 1$ is named typically "canonical system" or "canonical plant".

The following theorems and proofs are just discussed about the second order systems, but the conclusions are also correct for the nth order system.

Theorem 2.1: The maximum sampling-period, denoted as $\tau_{\max}$ within the permit of the tracking estimation error ε demanded in practice, of the LADRC based controller is depended on the plant's inherent "time-scaling", defined as ρ in definition 2.1.

Proof: A typical second order plant, represented by the state space equation, can be described as:

$$\begin{cases} \dot{X}(t) = AX(t) + B(u(t) + g(X,t)) + E\dot{f}(t) \\ y(t) = CX(t) \end{cases}, \tag{6}$$

where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $X(t) = \begin{bmatrix} x_1(t) \; x_2(t) \; x_3(t) \end{bmatrix}^T$ and $\dot{X}(t) = \begin{bmatrix} \dot{x}_1(t) \; \dot{x}_2(t) \; \dot{x}_3(t) \end{bmatrix}^T$ is the state vector and its derivative, respectively. And $g(X,t)$ is the known part of the plant's model, and $\dot{f}(t)$ is the derivative of $f(X,w,t)$, the uncertain function, which is bounded and continuous or piecewise continuous during an interval $0 \le t \le T$ for any $T > 0$.

The LESO of the above plant (6), according to preceding description, is constructed as

$$\begin{cases} \dot{Z}(t) = AZ(t) + B(u(t) + g(X,t)) + L(y(t) - \hat{y}(t)) \\ \hat{y}(t) = CZ(t) \end{cases}, \tag{7}$$

where $L = \begin{bmatrix} 3\omega_0 \; 3\omega_0^2 \; \omega_0^3 \end{bmatrix}^T$, is the observer gain vector, $Z = \begin{bmatrix} z_1 \; z_2 \; z_3 \end{bmatrix}^T$ is the estimate vector of X.

Then, subtracting (6) from (7) on both sides at the same time, the dynamics estimation error is obtained as:

$$d\dot{E}(t) = (A - LC)dE(t) + E\dot{f}(t), \tag{8}$$

where $dE(t) = \begin{bmatrix} \varepsilon_1 \; \varepsilon_2 \; \varepsilon_3 \end{bmatrix}^T$, $d\dot{E}(t) = \begin{bmatrix} \dot{\varepsilon}_1 \; \dot{\varepsilon}_2 \; \dot{\varepsilon}_3 \end{bmatrix}^T$, and $\varepsilon_i = x_i(t) - z_i(t), i = 1,2,3$.

Assume the sampling-period is τ, the discrete form of (8) becomes as:

$$dE(k)=(I+(A-LC)\tau)dE(k-1)+E\dot{f}(k-1)=$$
$$\left(I+((A-LC)\tau)^k dE(0)+\sum_{i=1}^{k}(I+(A-LC)\tau)^{k-i}E\dot{f}(k-1)\tau\right), \quad (9)$$
$$=\left(I+((A-LC)\tau)^k dE(0)+\sum H^{k-i}E\dot{f}(k-1)\tau\right)$$

where $(I+((A-LC)\tau)=\begin{bmatrix}1-3\omega_0\tau & \tau & 0\\ -3\omega_0^2\tau & 1 & \tau\\ -\omega_0^3\tau & 0 & 1\end{bmatrix}$ and $H^i=(I+((A-LC)\tau)^i E, i=1,2,\cdots,k$.

Assuming that $z_i(0)=x_i(0), i=1,2,3$, i.e. those initial values are equal, at the beginning of the LADRC based controller acting.

Let $\omega_0=m/\tau$, where $0<m<1$ is proposed [6], then the vector term H^i can be computed as:

$$H^i=\begin{bmatrix}\frac{\tau^2}{2}i(i-1)(1-m)^{i-2}\\ \tau i((i-2)m+1)(1-m)^{i-2}\\ ((i-1)(i-2)m^2+2(i-2)m+2)\frac{(1-m)^{i-2}}{2}\end{bmatrix}. \quad (10)$$

Having an insight into (10) and (9), it is obvious that the tracking estimation error, $\varepsilon_1=x_1-z_1$, is obtained as follow:

$$\begin{aligned}\varepsilon_1&=\sum_{i=1}^{k}H^{k-i}E\dot{f}(k-1)\tau\\ &=\sum_{i=1}^{k}\frac{\tau^3(k-i-1)(1-m)^{k-i-2}(k-i)}{2}\dot{f}(k-1)\\ &=\sum_{i=0}^{k-1}\frac{\tau^3(i-1)(1-m)^{i-2}i}{2}\dot{f}(k-1)\\ &\le\frac{\tau^2}{m^3}(f(k)-f(k-1))\\ &\le\frac{M\tau^2}{m^3}\end{aligned}. \quad (11)$$

So as long as the design requirements error ε satisfies $\frac{M\tau^2}{m^3}\le\varepsilon$ in practical engineering, then the actual tracking estimation error ε_1 will meet the design requirement. As a result of the above analysis, the following inequality expression is yielded as

$$\tau\le\sqrt{\frac{\varepsilon m^3}{M}}=\sqrt{\varepsilon m^3}\rho \quad , \quad (12)$$

which can be transformed into

$$\tau\le\tau_{\max}, \quad (13)$$

where $\tau_{\max}=\sqrt{\varepsilon m^3}\rho$. That is to say the "time-scaling" of the specific plant demands that the sampling-period of controller must be smaller than the $\tau_{\max}$, determined by (13), to meet the desired design allowable error ε.

Remark 2.1: Those plants, whose "time-scaling" are larger than the minimum "time-scaling" $\rho_{\min}$ which the controllers possess, can be controlled well by these LADRC based controllers.

Proof: The proof is obvious, form (12), the following inequality expression can be derived $\rho\ge\frac{\tau}{\sqrt{\varepsilon m^3}}$, which can also be converted as:

$$\rho\ge\rho_{\min}, \quad (14)$$

where $\rho_{\min}=\frac{\tau}{\sqrt{\varepsilon m^3}}$, which is decided by the sampling-period and the allowable tracking error of the controller. That is to say, the specific sampling-period τ and the demanded the tracking estimation error of the controller is given, then the disturbance rejection ability of this LADRC based controller is determined. In other words, the maximum amplitude of the plant's unknown disturbance is $\frac{1}{\rho_{\min}^2}$ for second plant.

3 Simulations

Assume there is a plant, which is a second-order system and described in state space as follow:

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=f(x_1,x_2,t)-100.4x_2+230u\\ y=x_1\end{cases}, \quad (15)$$

where $f(x_1,x_2,t)$ is the unknown disturbance excluding the known part of the plant's model. From previous analysis, it is known that its corresponding modified LESO is:

$$\begin{cases}e=y-z_1\\ \dot{z}_1=z_2+3\omega_0 e\\ \dot{z}_2=z_3+3\omega_0^2 e-100.4z_2+230u\\ \dot{z}_3=\beta_3\omega_0^3\\ y=x_1\end{cases}, \quad (16)$$

where, we adopt the sampling-period $\tau=1ms$, the observer bandwidth $\omega_0=300$ and $u=0$ respectively. Let us investigate the performance of above specified LESO on the various plants with different "time-scaling". Assuming there are three plants with different unknown disturbance $f(x_1, x_2, t)$, which are designated as the following three various expression:

A. $f(x_1, x_2, t) = 100\sin(6.28t)$,

B. $f(x_1, x_2, t) = 1000\sin(6.28t)$,

C. $f(x_1, x_2, t) = 4000\sin(6.28t)$,

which are regarded as three various plants whose "time-scaling" are $\rho_1 = 0.1$, $\rho_2 = 0.0316$ and $\rho_3 = 0.0158$ respectively. The simulation results are shown in figure 1 with the results of the plants A, B and C corresponding to subgraph (a), (b) and (c) respectively. And subgraph (d) in figure 1 shows the tracking errors along with the output y of the above three plants.

The subgraph (d) in figure 1 demonstrates that the tracking error is the largest on the plant C (the green curve in subgraph (d)) and that is the smallest on the plant A (the red curve in subgraph (d)). That to say, the larger the "time-scaling" of plant, the smaller of the tracking error is for the same LESO.

Suppose that the demanded tracking estimate error, in practice, is less than 0.02, then the above LESO can well track those plants whose "time-scaling" are larger than 0.043, i.e. $\rho \geq \rho_{\min} \approx 0.043$, which can be obtained from inequality expression (14).

On the other hand, figure 2 shows the simulation results about the LESO with different sampling-periods acting on the same plant A expressed above. The sampling-period of LESO, shown in figure 2, is $\tau_1 = 1ms$ in subgraph (a), $\tau_2 = 2ms$ in subgraph (b), $\tau_3 = 4ms$ in subgraph (c), and the three tracking errors are displayed in subgraph (d) in figure 2. The subgraph (d) in figure 2 demonstrates that the tracking error is the smallest with sampling-period $\tau_1 = 1ms$ (the red curve in subgraph (d)) and in contrast, that is the largest with sampling-period $\tau_3 = 4ms$ (the green curve in subgraph (d)). In other words, the smaller the sampling-period of LESO is, the higher the tracking precision is without considering the sampling noise. Assuming the required tracking error is $\varepsilon \leq 0.02$, this demands the maximum sampling-period of LESO is $\tau_{\max} \approx 2.3ms$ obtained from the inequality expression (13) to track the state variables well.

4 Conclusions

In this paper, the tracking performance of the LESO is discussed by the quantitative description. It is first time that the maximum sampling-period of LESO is presented by a specified formula (13) for an existing plant. On the other hand, the tracking ability of the designed LESO is presented by formula (14). The theory analysis in section II and simulation results in section III both show that the smaller the plant's "time-scaling", the larger tracking error is, using the same LESO. Conversely, the shorter of the sampling-period of LESO, the higher the tracking accuracy is for the same plant. The above discussions and analysis, especially the quantitative formulas (13) and (14) will help the scholars and engineers to design the proper LESO in practice.

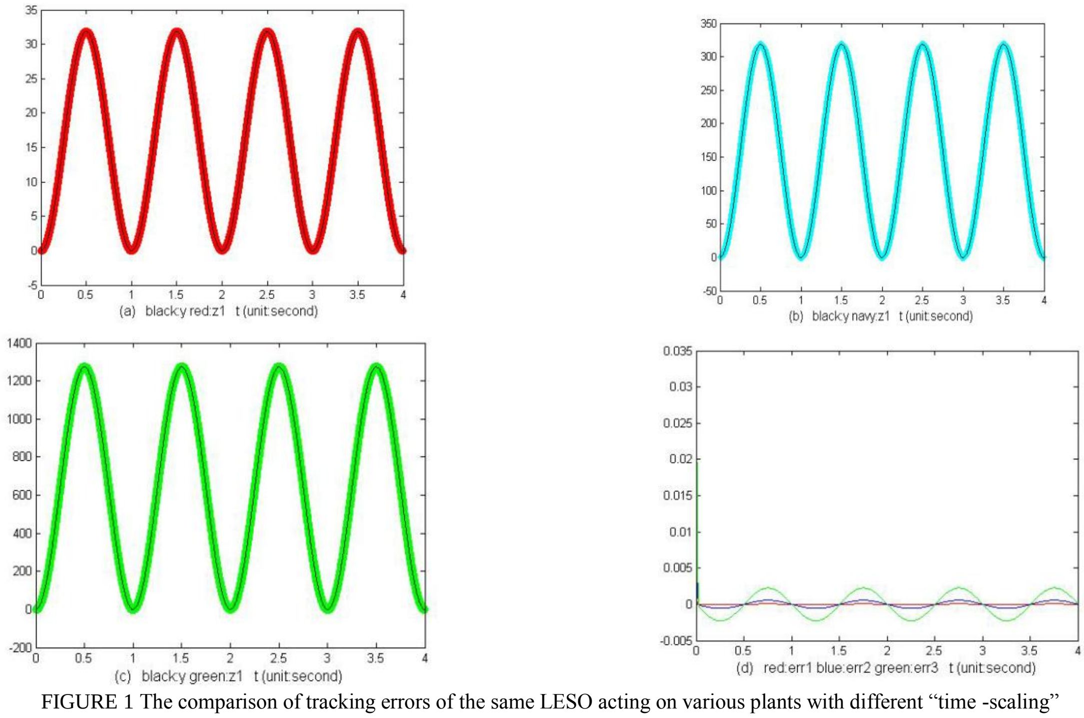

FIGURE 1 The comparison of tracking errors of the same LESO acting on various plants with different "time -scaling"

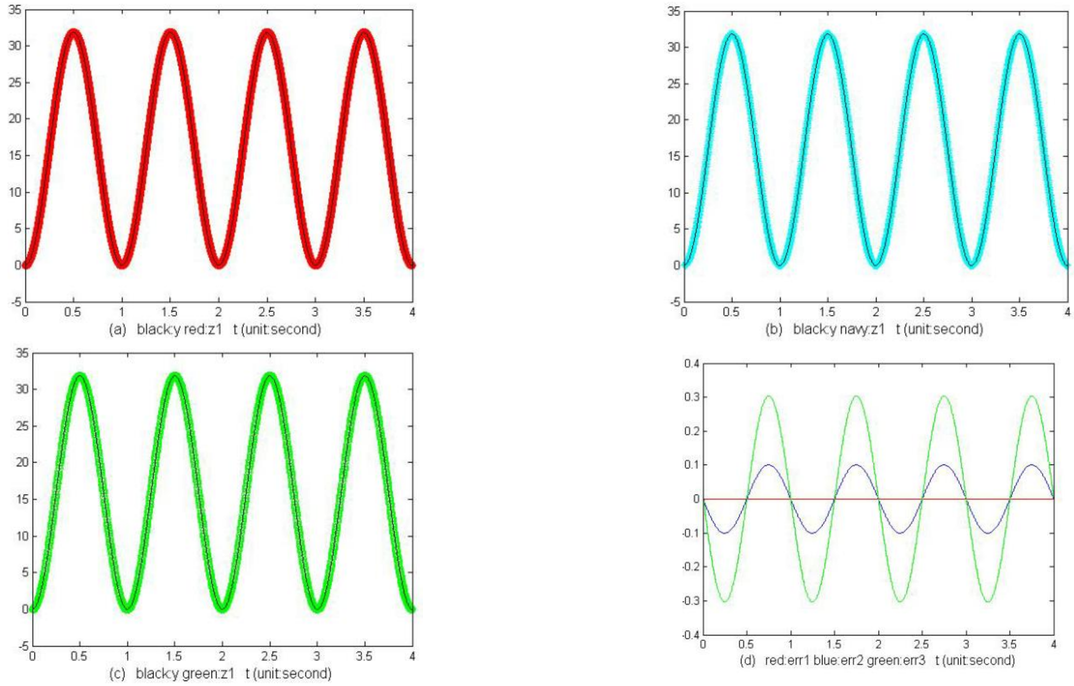

FIGURE 2 The comparison of tracking errors of the LESO with different sampling-period acting on the same plant

References

[1] Gao Z 2003 Scaling and Bandwidth-Parameterization Based Controller Tuning *In Proc. of the 2003 America Control Conference, Denver, CO, USA. June 2003* 4989-4996

[2] Bao Zhu Guo, Zhi Liang Zhao 2011 On the convergence of an extended state observer for nonlinear systems with uncertainty. *Systems & Control Letters* **60** 420–430

[3] Guo B Z, Zhao Z L 2012 On convergence of non-linear extended state observer for multi-input multi-output systems with uncertainty *IET Control Theory and Applications* **6**(15) 2375–2386

[4] Wenchao Xue, Yi Huang 2013 On Frequency-domain Analysis of ADRC for Uncertain System *In Proc. of 2013 American Control Conference, Washington, DC, USA, June 17-19, 2013*

[5] Han J 1994 The time scale and the nonlinear PID controller *In Proc. of the 1994 Chinese Control Conference, Taiyuan, China. August 15-22, 1994*

[6] Yoo D 2007 Optimal fast tracking observer bandwidth of the linear extended state observer *International Journal of Control* **80**(1) 102–111

Authors

Shangyao Shi

Current position, grades: Chief Engineer in Wuhan HuaGong Laser Engineering corporation, China.
University studies: received his Master of Engineering in Manufacture of Machinery and Automation (1997) and now is studying at Sichuan University with a major of Measurement and Control Technology and Instrumentation Program for PhD (2014).
Scientific interest: current research interests include Servo Control, Embedded System, Artificial Intelligence and Distributed Control Systems.

Shiping Zhao

Current position, grades: full professor of manufacturing science and engineering Department, Sichuan University.
University studies: received his PhD in engineering precision instruments and machinery (1991) from Chongqing University.
Scientific interest: He makes a great deal research on robot technology and non-destructive test.

Jun Li

University studies: M.Sc. in Engineering (2008) from southwest University of science and technology. Now he is studying at Sichuan University with a major of Mechatronic Engineering Program for PhD (2014).
Scientific interest: the Electrical discharge machining, Micro Electrical discharge machining and Artificial intelligence used in the mechanical engineering.