A few expanding integrable models of WKI hierarchy and their Hamiltonian structures

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Received 1 October 2014, www.cmnt.lv

Abstract

The integrable coupling of the WKI hierarchy is obtained by the perturbation approach. With the help of a higher dimensional loop algebra, the coupling integrable couplings of the WKI hierarchy are obtained, respectively. Their Hamiltonian structures are worked out by employing the component-trace identities and variational identity.

Keywords: coupling integrable couplings, component-trace identities, perturbation equation

1 Introduction

The notion on integrable couplings was introduced when study of Virasoro symmetric algebras [1, 2]. To find as many new integrable systems and their integrable couplings as possible and to elucidate in depth their algebraic and geometric properties are of both theoretical and practical value. During the past few years, some interesting integrable couplings and associated properties of some known interesting integrable hierarchies, such as the AKNS hierarchy, the KN hierarchy, the Burger hierarchy, etc. were obtained [3-13] In order to get Hamiltonian structures of integrable couplings, Guo and Zhang proposed the quadratic-form identity [14]. After this, Ma and Chen [15, 16] built the variational identity and generalized the quadratic-form identity and obtained some integrable couplings and their Hamiltonian structures. Recently, Ma and Zhang [17] proposed the notion on component-trace identities. They are very effective to construct the Hamiltonian structure of the perturbation equation. In Ref. [18], Ma and Gao proposed the notion called coupling integrable couplings of the nonlinear Schr¨odinger equation and associated symmetry properties, etc. Based on this, Zhang and Tam [19] constructed a few higher dimensional Lie algebras to obtain the coupling integrable couplings of the AKNS hierarchy and the KN hierarchy.

In the paper, we first give the first-order perturbation equation of the WKI hierarchy and its Hamiltonian structure is worked out by employing the component-trace identities. Then we use the way presented in Ref. [18-19] to investigate the coupling integrable couplings of the WKI hierarchy. In Refs. [18-19], the author didn't obtain the Hamiltonian structure of the coupling integrable couplings, while in the paper the Hamiltonian structures of the coupling integrable couplings of the WKI hierarchy will be worked out by using the variational identity.

2 The perturbation equation of the WKI hierarchy and its Hamiltonian structure

Yao and Zhang [20] utilized Tu scheme to obtain the multi-component WKI hierarchy. In this section, we take the perturbation way to deduce the integrable coupling of the WKI hierarchy and employ the component-trace identities to generate its Hamiltonian structure.

Consider the isospectral problem of the WKI hierarchy

$$\varphi_x = U\varphi, U = \begin{pmatrix} -i\lambda & u_1\lambda \\ u_2\lambda & i\lambda \end{pmatrix}, \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad (1)$$

where u_1, u_2 are potentials and λ is the spectral parameter. By means of constructing a proper time evolution

$$\varphi_{t_n} = V^{[n]}\varphi, V^{[n]} = \sum_{m=0}^{n} \begin{pmatrix} \lambda a_m & b_{mx} + i\lambda u_1 a_m \\ c_{mx} + i\lambda u_2 a_m & -\lambda a_m \end{pmatrix} \lambda^{n-m}, \quad (2)$$

and using the zero-curvature equation, we have the WKI hierarchy:

$$u_{t_n} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{t_n} = \begin{pmatrix} 0 & -\partial^2 \\ \partial^2 & 0 \end{pmatrix} L^n \begin{pmatrix} \frac{u_2}{\sqrt{1-u_1u_2}} \\ \frac{u_1}{\sqrt{1-u_1u_2}} \end{pmatrix} = JL^n \begin{pmatrix} \frac{u_2}{\sqrt{1-u_1u_2}} \\ \frac{u_1}{\sqrt{1-u_1u_2}} \end{pmatrix}, \quad (3)$$

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where:

$$L=\begin{pmatrix} -\frac{i}{2}\partial-\frac{i}{4}\frac{u_2}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_1}{\sqrt{1-u_1u_2}}\partial^2 & \frac{i}{4}\frac{u_2}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_2}{\sqrt{1-u_1u_2}}\partial^2 \\ -\frac{i}{4}\frac{u_1}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_1}{\sqrt{1-u_1u_2}}\partial^2 & \frac{i}{2}\partial+\frac{i}{4}\frac{u_1}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_2}{\sqrt{1-u_1u_2}}\partial^2 \end{pmatrix}. \quad (4)$$

Accordingly, when $n=0$, we can get the WKI equation. By using the race identity, we can obtain the Hamiltonian structure of the WKI hierarchy

$$u_{t_n}=J\frac{\delta H_{n-1}}{\delta u}, \quad (5)$$

where $H_{n-1}=\int\frac{a_{n-1}+u_2b_{n-1}-u_1c_{n-1}}{n-1}dx.$

Next, we will construct a integrable coupling of the WKI hierarchy by the perturbation approach and its Hamiltonian structure by the component-trace identities.

Let us take a matrix Lie algebra g consisting of the following matrices:

$$A=\begin{bmatrix} A_0 & A_1 & \cdots & \cdots & A_N \\ & A_0 & A_1 & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & A_0 & A_1 \\ 0 & & & & A_0 \end{bmatrix}, \quad (6)$$

where A_i, $0\le i\le N$, are square matrices of the same order. For convenience, we rewrite an element of the Lie algebra g as a vector of matrices: $A=(A_0,A_1,\cdots,A_N)$, where the components $A_i, 1\le i\le N$, are defined by:

$$A_i=\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}|_{\varepsilon=0}A(\hat{u}_N),\hat{u}_N=u+u+\sum_{i=1}^{N}\varepsilon^i\eta_i,1\le i\le N. \quad (7)$$

The enlarged zero-curvature equation by perturbation

$$\hat{U}_{Nt}-\hat{V}_{Nt}+[\hat{U}_N,\hat{V}_N]=0,$$

where $\hat{U}_N=u+\sum_{i=0}^{N}\varepsilon^i\eta_i$, $\hat{V}_N=V+\sum_{i=0}^{N}\varepsilon^i\eta_i, 1\le i\le N$ give rise to the perturbation equation of the Nth order:

$$\hat{\eta}_{Nt}=\hat{K}_N(\hat{\eta}_N)=(K^T(u),\frac{1}{1!}\frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}K^T(\hat{u}_N),\cdots,$$
$$\frac{1}{N!}\frac{\partial^N}{\partial\varepsilon^N}|_{\varepsilon=0}K^T(\hat{u}_N))^T,$$

where the column vector $\hat{\eta}_N$ of dependent variables is $\hat{\eta}_N=(u^T,\eta_1^T,\cdots,\eta_N^T)$. In what follows, we focus on the perturbation equation of the first order. We consider an isospectral problem as follows:

$$\bar{U}(\bar{u},\lambda)=\hat{U}_1=\begin{bmatrix} U_0 & U_1 \\ 0 & U_0 \end{bmatrix},\bar{V}(\bar{v},\lambda)=\hat{V}_1=\begin{bmatrix} V_0 & V_1 \\ 0 & V_0 \end{bmatrix}, \quad (8)$$

where U_0 and V_0 are defined by Equations (1) and (2), U_1 and V_1 are showed as follows:

$$U_1=\frac{1}{1!}\frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}U(\hat{u}_1)=\frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}\begin{bmatrix} -i\lambda & u_1\lambda+\varepsilon\lambda u_3 \\ u_2\lambda+\varepsilon\lambda u_4 & i\lambda \end{bmatrix}=$$
$$\begin{bmatrix} 0 & \lambda u_3 \\ \lambda u_4 & 0 \end{bmatrix}, \quad (9)$$

$$V_1=\frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}\begin{bmatrix} \lambda a+\varepsilon\lambda d & B \\ C & -\lambda a-\varepsilon\lambda d \end{bmatrix}=$$
$$\begin{bmatrix} \lambda d & f_x+i\lambda u_4a+i\lambda u_1d \\ g_x+i\lambda u_5a+i\lambda u_2d & -\lambda d \end{bmatrix}, \quad (10)$$

where:

$$B=b_x+i\lambda u_1a+\varepsilon(f_x+i\lambda u_4a+i\lambda u_1d),$$
$$C=c_x+i\lambda u_2a+\varepsilon(g_x+i\lambda u_5a+i\lambda u_2d).$$

The enlarged stationary zero-curvature equation $\hat{V}_x=[\hat{U},\hat{V}]$ equivalently yields:

$$\begin{cases} V_x=[U,V], \\ V_{1x}=[U,V_1]+[U_1,V]. \end{cases} \quad (11)$$

A direct calculation leads to:

$$\begin{cases} a_{mx} = u_1 c_{mx} - u_2 b_{mx}, \\ i(u_1 a_{m+1})_x + b_{mxx} = -2ib_{m+1x}, \\ i(u_2 a_{m+1})_x + c_{mxx} = 2ic_{m+1x}, \\ d_{mx} = u_3 c_{mx} + u_1 f_{mx} - u_2 e_{mx} - u_4 b_{mx}, \\ f_{mxx} + i(u_1 d_{m+1})_x + i(u_3 a_{m+1})_x = -2if_{m+1x}, \\ g_{mxx} + i(u_2 d_{m+1})_x + i(u_4 a_{m+1})_x = 2ig_{m+1x}, \\ a_0 = \alpha_1, b_0 = \alpha_2, c_0 = \alpha_3, d_0 = \alpha_4, g_0 = \alpha_5, f_0 = \alpha_6, \\ a_1 = \dfrac{2}{\sqrt{1-u_1u_2}}, b_1 = \dfrac{-u_1}{\sqrt{1-u_1u_2}}, c_1 = \dfrac{u_2}{\sqrt{1-u_1u_2}}, \\ d_1 = \dfrac{1}{\sqrt{1-u_1u_2}} + \dfrac{u_1u_4 + u_2u_3}{(1-u_1u_2)^{3/2}}, \\ f_1 = -\dfrac{u_1}{2\sqrt{1-u_1u_2}} - \dfrac{u_1(u_1u_4+u_2u_3)}{2(1-u_1u_2)^{3/2}} - \dfrac{u_3}{\sqrt{1-u_1u_2}}, \\ g_1 = \dfrac{u_2}{2\sqrt{1-u_1u_2}} + \dfrac{u_2(u_1u_4+u_2u_3)}{2(1-u_1u_2)^{3/2}} + \dfrac{u_4}{\sqrt{1-u_1u_2}}. \end{cases} \quad (12)$$

From the recursion relation in Equation (12), we have a recursive formula for determining f_n, g_n:

$$\begin{bmatrix} g_n \\ -f_n \end{bmatrix} = L \begin{bmatrix} g_{n-1} \\ -f_{n-1} \end{bmatrix} + L_1 \begin{bmatrix} c_{n-1} \\ -b_{n-1} \end{bmatrix} = L \begin{bmatrix} g_{n-1} \\ -f_{n-1} \end{bmatrix} + \begin{bmatrix} L_{11} & L_{12} \\ L_{13} & L_{14} \end{bmatrix} \begin{bmatrix} c_{n-1} \\ -b_{n-1} \end{bmatrix}, \quad (13)$$

where:

$$L_{11} = -\frac{i}{4}\frac{u_4}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_1}{\sqrt{1-u_1u_2}}\partial^2 - \frac{i}{8}\frac{u_2}{\sqrt{1-u_1u_2}}\partial^{-1}\left(\frac{u_1}{\sqrt{1-u_1u_2}}\partial\frac{u_4}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_1}{\sqrt{1-u_1u_2}}\partial^2 + \frac{u_2}{\sqrt{1-u_1u_2}}\partial\frac{u_3}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_1}{\sqrt{1-u_1u_2}}\partial^2 + \frac{u_3}{\sqrt{1-u_1u_2}}\partial\frac{u_2}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_1}{\sqrt{1-u_1u_2}}\partial^2 + \frac{u_4}{\sqrt{1-u_1u_2}}\partial\frac{u_1}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_1}{\sqrt{1-u_1u_2}}\partial^2 + 2i\frac{u_3}{\sqrt{1-u_1u_2}}\partial^2\right),$$

$$L_{12} = \frac{i}{4}\frac{u_4}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_2}{\sqrt{1-u_1u_2}}\partial^2 + \frac{i}{8}\frac{u_2}{\sqrt{1-u_1u_2}}\partial^{-1}\left(\frac{u_1}{\sqrt{1-u_1u_2}}\partial\frac{u_4}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_2}{\sqrt{1-u_1u_2}}\partial^2 + \frac{u_2}{\sqrt{1-u_1u_2}}\partial\frac{u_3}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_2}{\sqrt{1-u_1u_2}}\partial^2 + \frac{u_3}{\sqrt{1-u_1u_2}}\partial\frac{u_2}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_3}{\sqrt{1-u_1u_2}}\partial^2 + \frac{u_4}{\sqrt{1-u_1u_2}}\partial\frac{u_1}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_2}{\sqrt{1-u_1u_2}}\partial^2 + 2i\frac{u_4}{\sqrt{1-u_1u_2}}\partial^2\right),$$

$$L_{13} = -\frac{i}{4}\frac{u_3}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_1}{\sqrt{1-u_1u_2}}\partial^2 - \frac{i}{8}\frac{u_1}{\sqrt{1-u_1u_2}}\partial^{-1}\left(\frac{u_1}{\sqrt{1-u_1u_2}}\partial\frac{u_4}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_1}{\sqrt{1-u_1u_2}}\partial^2 + \frac{u_2}{\sqrt{1-u_1u_2}}\partial\frac{u_3}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_1}{\sqrt{1-u_1u_2}}\partial^2 + \frac{u_3}{\sqrt{1-u_1u_2}}\partial\frac{u_2}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_1}{\sqrt{1-u_1u_2}}\partial^2 + \frac{u_4}{\sqrt{1-u_1u_2}}\partial\frac{u_1}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_1}{\sqrt{1-u_1u_2}}\partial^2 + 2i\frac{u_3}{\sqrt{1-u_1u_2}}\partial^2\right),$$

$$L_{14} = \frac{i}{4}\frac{u_3}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_2}{\sqrt{1-u_1u_2}}\partial^2 + \frac{i}{8}\frac{u_1}{\sqrt{1-u_1u_2}}\partial^{-1}\left(\frac{u_1}{\sqrt{1-u_1u_2}}\partial\frac{u_4}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_2}{\sqrt{1-u_1u_2}}\partial^2 + \frac{u_2}{\sqrt{1-u_1u_2}}\partial\frac{u_3}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_2}{\sqrt{1-u_1u_2}}\partial^2 + \frac{u_3}{\sqrt{1-u_1u_2}}\partial\frac{u_2}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_3}{\sqrt{1-u_1u_2}}\partial^2 + \frac{u_4}{\sqrt{1-u_1u_2}}\partial\frac{u_1}{\sqrt{1-u_1u_2}}\partial^{-1}\frac{u_2}{\sqrt{1-u_1u_2}}\partial^2 + 2i\frac{u_4}{\sqrt{1-u_1u_2}}\partial^2\right).$$

Then the enlarged zero-curvature equation $\hat{U}_{t_n}-\hat{V}_x^{[n]}+[\hat{U},\hat{V}^{[n]}]=0$ yields the hierarchy of the first-order perturbation equation:

$$\bar{U}_{t_n}=(u_1,u_2,u_3,u_4)_{t_n}^T=\begin{bmatrix}0 & J\\ J & J_1\end{bmatrix}\left(g_{n-1},-f_{n-1},c_{n-1},-b_{n-1}{}^T\right), \quad (14)$$

where

$$J_1=\frac{1}{1!}\frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}\ J(\hat{u}_1)=\begin{bmatrix}0 & -\partial^2\\ \partial^2 & 0\end{bmatrix}. \quad (15)$$

Hence, we can get the following integrable couplings of Equation (3) as follows:

$$\begin{cases} u_{1t}=\left(\dfrac{iu_1}{\sqrt{1-u_1u_2}}\right)_{xx}, \\ u_{2t}=\left(-\dfrac{iu_2}{\sqrt{1-u_1u_2}}\right)_{xx}, \\ u_{3t}=\left(-\dfrac{u_1}{2\sqrt{1-u_1u_2}}-\dfrac{u_1(u_1u_4+u_2u_3)}{2(1-u_1u_2)^{\frac{3}{2}}}-\dfrac{u_3}{\sqrt{1-u_1u_2}}\right)_{xx}, \\ u_{4t}=\left(\dfrac{u_2}{2\sqrt{1-u_1u_2}}+\dfrac{u_2(u_1u_4+u_2u_3)}{2(1-u_1u_2)^{\frac{3}{2}}}+\dfrac{u_4}{\sqrt{1-u_1u_2}}\right)_{xx}. \end{cases} \quad (16)$$

In order to generate the Hamiltonian structure of the first-order perturbation equation of the WKI hierarchy, we introduce the following theorem:

Theorem 1 [17] Let g be a matrix Lie algebra consisting of block matrices defined by Equation (6). For a given spectral matrix $U=U(u,\lambda)=(U_0,U_1,\cdots,U_N)\in g$, we have the variational identity:

$$\frac{\delta}{\delta u}\int\sum_{k=0}^{N}\alpha_k\operatorname{tr}\left(\sum_{i+j=k}V_i\frac{\partial U_j}{\partial\lambda}\right)dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\sum_{k=0}^{N}\alpha_k\operatorname{tr}\left(\sum_{i+j=k}V_i\frac{\partial U_j}{\partial u}\right), \quad (17)$$

where $V=V(v,\lambda)=(V_0,V_1,\cdots,V_N)\in g$ satisfies the zero-curvature equation, all α_k's are arbitrary constants with $\alpha_N\neq 0$ and γ is the constant determined by $\gamma=-\frac{1}{2}\frac{d}{d\lambda}ln|<V,V>|$. This variational identity Equation (17) is called the component-trace identity. For a general integer N, we have:

$$\frac{\delta}{\delta u}\int\operatorname{tr}\left(\sum_{i+j=N}V_i\frac{\partial U_j}{\partial\lambda}\right)dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\operatorname{tr}\left(\sum_{i+j=N}V_i\frac{\partial U_j}{\partial u}\right), \quad (18)$$

which is called the last-component-trace identity. Then, the generating function of Hamiltonian functions for the perturbation equation of N-th order is computed as follows: $\hat{H}(\hat{\eta}_N)=\operatorname{tr}\left(\sum_{i+j=N}V_i\frac{\partial U_j}{\partial\lambda}\right)$, where $\hat{\eta}_N=(u^T,\eta_1^T,\cdots,\eta_N^T)^T$. This implies that the last-component-trace identity provides the generation function of Hamiltonian functions for the perturbation equation. Then, basing on the generating function of Hamiltonian functions for the original equation $H_g(u)=\operatorname{tr}\left(V\frac{\partial U}{\partial\lambda}\right)$, we can get the generating function of Hamiltonian functions for the perturbation equations of N-th order as follows:

$$\begin{aligned}&\hat{H}_{g,N}(\hat{\eta}_N)=\frac{1}{N!}\frac{\partial^N}{\partial\varepsilon^N}|_{\varepsilon=0}\ H_g(\hat{u}_N)=\\ &\operatorname{tr}\left(\frac{1}{N!}\frac{\partial^N}{\partial\varepsilon^N}|_{\varepsilon=0}\ V(\hat{u}_N)\frac{\partial}{\partial\lambda}U(\hat{u}_N)\right)=\\ &\operatorname{tr}\left(\frac{1}{N!}\sum_{i+j=N}\binom{i}{N}\frac{\partial^i}{\partial\varepsilon^i}|_{\varepsilon=0}\ V(\hat{u}_N)\frac{\partial^j}{\partial\varepsilon^j}|_{\varepsilon=0}\ \frac{\partial}{\partial\lambda}U(\hat{u}_N)\right)=\\ &\operatorname{tr}\left(\sum_{i+j=N}\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}|_{\varepsilon=0}\ V(\hat{u}_N)\frac{\partial}{\partial\lambda}(\frac{1}{j!}\frac{\partial^j}{\partial\varepsilon^j}|_{\varepsilon=0}\ U(\hat{u}_N)\right)=\\ &\operatorname{tr}\left(\sum_{i+j=N}V_i\frac{\partial U_j}{\partial\lambda}\right).\end{aligned} \quad (19)$$

By using the above results, a direct calculation reads

$$\begin{aligned}&\operatorname{tr}\left(\sum_{i+j=1}V_i\frac{\partial U_j}{\partial\lambda}\right)=\operatorname{tr}\left(V_1\frac{\partial U_0}{\partial\lambda}+V_0\frac{\partial U_1}{\partial\lambda}\right)=\\ &(-2i\lambda d-2iu_2f+2iu_1g-2iu_4b+2iu_3c),\end{aligned} \quad (20)$$

$$\begin{aligned}&\operatorname{tr}(V_1\frac{\partial U_0}{\partial u_1})=2i\lambda g,\operatorname{tr}\left(V_1\frac{\partial U_0}{\partial u_2}\right)=-2i\lambda f,\\ &\operatorname{tr}\left(V_0\frac{\partial U_1}{\partial u_3}\right)=2i\lambda c,\operatorname{tr}\left(V_0\frac{\partial U_1}{\partial u_4}\right)=-2i\lambda b.\end{aligned} \quad (21)$$

Basing on the last-component-trace identities, we have

$$\begin{aligned}&\frac{\delta}{\delta u}\int(-2id_{n-1}-2iu_2f_{n-1}+2iu_1g_{n-1}-\\ &2iu_4b_{n-1}+2iu_3c_{n-1})dx=2i(2+\gamma-n)[g,-f,c,-b]^T.\end{aligned} \quad (22)$$

Take $n=2$ in above equation gives $\gamma=-1$. Thus, the Hamiltonian structure of the perturbation equation of the WKI hierarchy is derived as follows:

$$\hat{U}_{t_n}=(u_1,u_2,u_3,u_4)_{t_n}^T=\begin{bmatrix}0 & J\\ J & J_1\end{bmatrix}\frac{\delta H_{n-1}}{\delta u}, \quad (23)$$

where:

$$H_{n-1}=\frac{1}{n-1}\int(d_{n_1}+u_2f_{n-1}-u_1g_{n-1}+u_4b_{n-1}-u_3c_{n-1})dx.$$

3 The coupling integrable couplings of the WKI hierarchy and its Hamiltonian structure

The coupling of the WKI hierarchy is given in the above. In the section, we will construct the coupling integrable couplings by following the way in Ref. [18], which is introduced as follows.

Given two integrable couplings of the integrable equation $u_t = K(u)$:

$$\bar{u}_{1,t} = \bar{K}_1(\overline{u_1}) = \begin{bmatrix} K(u) \\ S(u,v) \end{bmatrix}, \bar{u}_1 = \begin{bmatrix} u \\ v \end{bmatrix}, \quad (24)$$

$$\bar{u}_{2,t} = \bar{K}_2(\overline{u_2}) = \begin{bmatrix} K(u) \\ T(u,v) \end{bmatrix}, \bar{u}_2 = \begin{bmatrix} u \\ w \end{bmatrix}, \quad (25)$$

a new bigger system is formed as follows:

$$\hat{u}_t = \hat{K}(\hat{u}) = \begin{bmatrix} K(u) \\ S(u,v) \\ T(u,v) \end{bmatrix}, \hat{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}. \quad (26)$$

We call Equation (26) coupling integrable couplings of Equations (24) and (25).

First, we will construct a 9-dimensional vector-Lie algebra and its corresponding loop algebra. Consider a vector space [18]:

$$R^9 = span\{a = (a_1, \cdots, a_9)^T, a_i \in R, i = 1, 2, \cdots, 9\}.$$

For $\forall a = (a_1, \cdots, a_9)^T$, $b = (b_1, \cdots, b_9)^T$, define a commutation operation:

$$\begin{aligned}[][a,b] = (&a_2b_3 - a_3b_2, 2(a_1b_2 - a_2b_1), 2(a_3b_1 - a_1b_3), a_2b_6 - a_6b_2 + a_5b_3 - a_3b_5, 2(a_1b_5 - a_5b_1 + a_4b_2 - a_2b_4), \\ &2(a_6b_1 - a_1b_6 + a_3b_4 - a_4b_3), a_2b_9 - a_9b_2 + a_8b_3 - a_3b_8 + a_8b_9 - a_9b_8, 2(a_1b_8 - a_8b_1 + a_7b_2 - a_2b_7 + a_7b_8 - a_8b_7), \\ &2(a_9b_1 - a_1b_9 + a_3b_7 - a_7b_3 + a_9b_7 - a_7b_9))^T\end{aligned} \quad (27)$$

It is verified that R^9 is a Lie algebra if equipped with Equation (27). Take a basis of R^9 as follows:

$$e_i = (e_{i1}, \cdots, e_{i9})^T, e_{ij} = \begin{cases} 1, i = j, \\ 0, i \neq j, 1 \leq i, j \leq 9. \end{cases} \quad (28)$$

A loop algebra $\tilde{R}^9$ corresponding to the Lie algebra R^9 is defined as:

$$\tilde{R}^9 = span\{e_i(n)\}_{i=1}^9, e_i(n) = e_i\lambda^n, [e_i(m) - e_j(n)] = [e_i, e_j]\lambda^{m+n}, 1 \leq i, j \leq 9, m, n \in Z. \quad (29)$$

By employing the loop algebra $\tilde{R}^9$, we consider the following Lax pair:

$$\begin{cases} U = -ie_1(1) + u_1e_2(1) + u_2e_3(1) + u_3e_5(1) + u_3e_6(1) + u_5e_8(1) + u_6e_9(1), \\ V = \sum_{m=0}^{\infty}[a_me_1(1-m) + iu_1a_me_2(1-m) + b_{mx}e_2(-m) + iu_2a_me_3(1-m) + c_{mx}e_3(-m)n + \\ d_me_4(1-m) + f_{mx}e_5(-m) + i(u_3a_m + u_1d_m)e_5(1-m) + g_{mx}e_6(-m) + i(u_4a_m + u_2d_m)e_6(1-m) + \\ h_me_7(1-m) + p_{mx}e_8(-m) + i(u_5a_m + u_5h_m + u_1h_m)e_8(1-m) + q_{mx}e_9(-m)n + \\ i(u_6a_m + u_6h_m + u_2h_m)e_9(1-m)] \end{cases}.$$

A solution to $V_x = [U, V]$ exhibits that:

$$\begin{cases} a_{mx} = u_1c_{mx} - u_2b_{mx}, \\ i(u_1a_{m+1})_x + b_{mxx} = -2ib_{m+1x}, \\ i(u_2a_{m+1})_x + c_{mxx} = 2ic_{m+1x}, \\ d_{mx} = u_1g_{mx} - u_2f_{mx} + u_3c_{mx} - u_4b_{mx}, \\ i(u_3a_{m+1} + u_1d_{m+1})_x + f_{mxx} = -2if_{m+1x}, \\ i(u_4a_{m+1} + u_2d_{m+1})_x + g_{mxx} = 2ig_{m+1x}, \\ h_{mx} = u_1q_{mx} - u_2p_{mx} + u_5c_{mx} - u_6b_{mx} + u_5q_{mx} - u_6p_{mx}, \\ i(u_5a_{m+1} + u_5h_{m+1} + u_1h_{m+1})_x + p_{mxx} = -2ip_{m+1x}, \\ i(u_6a_{m+1} + u_6h_{m+1} + u_2h_{m+1})_x + q_{mxx} = 2iq_{m+1x}. \end{cases} \quad (30)$$

Note:

$$V_{+}^{(n)}=\sum_{m=0}^{n}[a_m e_1(n+1-m)+iu_1 a_m e_2(n+1-m)+b_{mx}e_2(n-m)+iu_2 a_m e_3(n+1-m)+$$
$$c_{mx}e_3(n-m)+d_m e_4(n+1-m)+f_{mx}e_5(n-m)+i(u_3 a_m+u_1 d_m)e_5(n+1-m)+g_{mx}e_6(n-m)+$$
$$i(u_4 a_m+u_2 d_m)e_6(n+1-m)+h_m e_7(n+1-m)+p_{mx}e_8(n-m)+i(u_5 a_m+u_5 h_m+u_1 h_m)e_8(n+1-m)+$$
$$q_{mx}e_9(n-m)+i(u_6 a_m+u_6 h_m+u_2 h_m)e_9(n+1-m)].$$

Set $V_{-}^{(n)}=\lambda^n V-V_{+}^{(n)}$, we have:

$$-V_{+x}^{(n)}+[U,V_{+}^{(n)}]=-\lambda b_{n-1xx}e_2(0)-\lambda c_{n-1xx}e_3(0)-\lambda f_{n-1xx}e_5(0)-\lambda g_{n-1xx}e_6(0)-\lambda p_{n-1xx}e_8(0)-\lambda q_{n-1xx}e_9(0). \quad (31)$$

Therefore, the zero-curvature equation

$$U_t-V_x^{(n)}+[U,V^{(n)}]=0, \quad (32)$$

admits the following bigger integrable system:

$$U_t=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\end{pmatrix}_t= \quad (33)$$
$$\begin{pmatrix}0&0&0&-\partial^2&0&0\\0&0&\partial^2&0&0&0\\0&-\partial^2&0&0&0&-\partial^2\\\partial^2&0&0&0&\partial^2&0\\0&0&0&-\partial^2&0&-\partial^2\\0&0&\partial^2&0&\partial^2&0\end{pmatrix}\begin{pmatrix}c_{n-1}+g_{n-1}+q_{n-1}\\-b_{n-1}-f_{n-1}-p_{n-1}\\c_{n-1}\\-b_{n-1}\\c_{n-1}+q_{n-1}\\-b_{n-1}-p_{n-1}\end{pmatrix}.$$

Substituting $u_5=u_6=0$ in Equation (33) reduces to the Equation (14), which is an integrable coupling of the WKI hierarchy; when taking $u_3=u_4=0$ in Equation (33) reduces to another integrable coupling of the WKI hierarchy. So we call Equation (33) the coupling integrable couplings of the WKI hierarchy.

In order to deduce to the Hamiltonian structure of Equation (33), we rewrite Equation (30) as follows:

$$[a,b]=a^T R(b), \quad (34)$$

where:

$$R(b)=\begin{pmatrix}0&2b_2&-2b_3&0&2b_5&-2b_6\ 0&2b_8&-2b_9&0\\b_3&-2b_1&0&b_6&-2b_4&0&b_9&-2b_7&0\\-b_2&0&2b_1&-b_5&0&2b_4&-b_8&0&2b_7\\0&0&0&0&2b_2&-2b_3&0&0&0\\0&0&0&b_3&-2b_1&0&0&0&0\\0&0&0&-b_2&0&2b_1&0&0&0\\0&0&0&0&0&0&0&2(b_2+b_8)&-2(b_3+b_9)\\0&0&0&0&0&0&b_3+b_9&-2(b_1+b_7)&0\\0&0&0&0&0&0&-(b_2+b_8)&0&2(b_1+b_7)\end{pmatrix}.$$

Solving the matrix equation for the constant matrix F:

$$R(b)F=-(R(b)F)^T, F^T=F,$$

$$F=\begin{pmatrix} 2&0&0&2&0&0&2&0&0\\ 0&0&1&0&0&1&0&0&1\\ 0&1&0&0&1&0&0&1&0\\ 2&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 2&0&0&0&0&0&2&0&0\\ 0&0&1&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&1&0 \end{pmatrix}. \quad (35)$$

Then in terms of F, define a linear functional: $\{a,b\}=a^T Fb.$ Rewrite the Lax pair as follows:

$$\begin{cases} U=(-i\lambda,\lambda u_1,\lambda u_2,0,\lambda u_3,\lambda u_4,0,\lambda u_5,\lambda u_6)^T,\\ V=(\lambda a,b_x+i\lambda u_1 a,c_x+i\lambda u_2 a,\lambda d,f_x+i\lambda(u_3 a+u_1 d),g_x+i\lambda(u_4 a+u_2 d),\lambda h,\\ p_x+i\lambda(u_5 a+u_5 h+u_1 h),q_x+i\lambda(u_6 a+u_5 h+u_1 h))^T. \end{cases} \quad (36)$$

By using the above linear functional, we have:

$$\left\{V,\frac{\partial U}{\partial\lambda}\right\}=-2i(a+d+h)+2iu_1(c+g+q)-2iu_2(b+f+p)+2iu_3c-2iu_4b+2iu_5(c+q)-2iu_6(b+p),$$

$$\left\{V,\frac{\partial U}{\partial u_1}\right\}=2i\lambda(c+g+q),\left\{V,\frac{\partial U}{\partial u_2}\right\}=-2i\lambda(b+f+p),\left\{V,\frac{\partial U}{\partial u_3}\right\}=2i\lambda c,\left\{V,\frac{\partial U}{\partial u_4}\right\}=-2i\lambda b,$$

$$\left\{V,\frac{\partial U}{\partial u_5}\right\}=2i\lambda(c+q),\left\{V,\frac{\partial U}{\partial u_6}\right\}=-2i\lambda(b+p).$$

According the variational identity, we have

$$\frac{\delta}{\delta u}\int(-2i(a+d+h)+2iu_1(c+g+q)-2iu_2(b+f+p)+2iu_3c-2iu_4b+2iu_5(c+q)-2iu_6(b+p))dx=$$
$$\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}(2i\lambda(c+g+q),-2i\lambda(b+f+p),2i\lambda c,-2i\lambda b,2i\lambda(c+q),-2i\lambda(b+p))^T.$$

Comparing the coefficients of λ^{-n+1} yields

$$\frac{\delta}{\delta u}\int(-2i(a_{n-1}+d_{n-1}+h_{n-1})+2iu_1(c_{n-1}+g_{n-1}+q_{n-1})-2iu_2(b_{n-1}+f_{n-1}+p_{n-1})+2iu_3c_{n-1}-$$
$$2iu_4b_{n-1}+2iu_5(c_{n-1}+q_{n-1})-2iu_6(b_{n-1}+p_{n-1}))dx=2i(2+\gamma-n)\begin{pmatrix}-c_{n-1}-g_{n-1}\\ b_{n-1}+f_{n-1}\\ -c_{n-1}\\ b_{n-1}\end{pmatrix}.$$

Taking $n=2$ in above equation gives $\gamma=-1.$
Hence, the coupling integrable couplings of WKI hierarchy Equation (33) can be written as a Hamiltonian form:

$$U_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}_t = \begin{pmatrix} 0 & 0 & 0 & -\partial^2 & 0 & 0 \\ 0 & 0 & \partial^2 & 0 & 0 & 0 \\ 0 & -\partial^2 & 0 & 0 & 0 & -\partial^2 \\ \partial^2 & 0 & 0 & 0 & \partial^2 & 0 \\ 0 & 0 & 0 & -\partial^2 & 0 & -\partial^2 \\ 0 & 0 & \partial^2 & 0 & \partial^2 & 0 \end{pmatrix} \begin{pmatrix} c_{n-1} + g_{n-1} + q_{n-1} \\ -b_{n-1} - f_{n-1} - p_{n-1} \\ c_{n-1} \\ -b_{n-1} \\ c_{n-1} + q_{n-1} \\ -b_{n-1} - p_{n-1} \end{pmatrix} = J \frac{\delta H_{n-1}}{\delta u}, \quad (37)$$

where

$$H_{n-1} = \frac{1}{n-1} \int ((a_{n-1} + d_{n-1} + h_{n-1}) - u_1(c_{n-1} + g_{n-1} + q_{n-1}) + u_2(b_{n-1} + f_{n-1} + p_{n-1}) -$$
$$u_3 c_{n-1} + u_4 b_{n-1} - u_5(c_{n-1} + q_{n-1}) + u_6(b_{n-1} + p_{n-1}))dx,$$

J is a Hamiltonian operator.

4 Conclusions

In Ref.[19], the coupling integrable couplings of KN and AKNS hierarchy are obtained, but their Hamiltonian structures aren't given. In the paper, however, the integrable coupling of WKI hierarchy is obtained by the perturbation approach and its Hamiltonian structure is given by using the component-trace identities. Meanwhile, basing on a 9-dimensional Lie algebra, we discuss the coupling integrable couplings of the WKI hierarchy and obtain its Hamiltonian structure by the variational identity.

In the future, we will discuss the perturbation equation and coupling integrable couplings of other hierarchies and their Hamiltonian structures.

Acknowledgements

This work was supported by Nature Science Foundation of Shandong Province of China (No.ZR2013AQ017), Open Fund of the Key Laboratory of Ocean Circulation and Waves, Chinese Academy of Science (No. KLOCAW1401).

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